

3 (Sem-3) MAT M 2

2017

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

GROUP—A

(Linear Algebra)

(Marks : 40)

1. Answer the following as directed : $1 \times 7 = 7$

(a) Let $U = \{(a, b, c) : a = b = c\}$ is a subset in \mathbb{R}^3 . Determine whether or not U is a subspace of \mathbb{R}^3 .

(b) If $P_n(t)$ be the vector space of all polynomials of degree $\leq n$, then $\dim P_n(t)$ is

- (i) $n - 1$ (ii) n
(iii) $n + 1$ (iv) n^2

(Choose the correct option)

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(Turn Over)

- (c) The set \mathbb{R} of all real numbers is a vector space over the field \mathbb{Q} of rational numbers. Examine whether or not the set $\{1, \sqrt{2}\}$ of vectors in \mathbb{R} is linearly independent.
- (d) Let $v = (1, 2, 3)$ and scalar $k = -3$, show that $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (|x|, y + z)$ is not linear.
- (e) If T is a linear operator, then the following are equivalent :
- A scalar λ is an eigenvalue of T
 - The linear operator $\lambda I - T$ is singular
- (Write true or false)
- (f) Let V be vector space over the field F and let T be a linear operator on V . Define the characteristic space associated with a characteristic value of T .
- (g) Prove that for a linear operator (matrix) T , the scalar 0 is an eigenvalue of T if and only if T is singular.

2. Answer the following questions : 2×4=8

- (a) Express $v = (2, -5, 3)$ in \mathbb{R}^3 as a linear combination of the vectors

$$u_1 = (1, -3, 2), u_2 = (2, -4, -1), u_3 = (1, -5, 7)$$

- (b) Let \mathbb{R} be the field of real numbers and V be the space of all functional from \mathbb{R} into \mathbb{R} which are continuous. Define T by

$$(Tf)(x) = \int_0^x f(t) dt$$

Show that T is a linear transformation from V into V .

- (c) Consider the two bases of the vector space $\mathbb{R}^2(\mathbb{R})$:

$$B_1 = \{(1, 2), (3, 5)\} \text{ and } B_2 = \{(1, -1), (1, -2)\}$$

Find the change-of-basis matrix M from B_1 to the 'new' basis B_2 .

- (d) Using Cayley-Hamilton theorem, compute the inverse of

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

3. Answer any one part :

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(a) Let V and W be vector spaces over the same field F and T be a linear transformation from V into W . Show that if V is finite dimensional, then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

(b) Let V and W be vector spaces over the same field F and the linear mapping $T: V \rightarrow W$ is one-to-one and onto. Show that the inverse map $T^{-1}: W \rightarrow V$ is also linear.

4. Answer the following questions :

10×2=20

(a) When a vector space is said to be finitely generated? If V is a finitely generated vector space over a field F , prove that V has a finite basis and any two bases of V have same number of vectors.

Or

Suppose V is finite dimensional vector space over a field F and U is a subspace of V . Prove that there is a subspace W of V such that $V = U \oplus W$.

(b) State and prove Cayley-Hamilton theorem for the characteristic polynomial f of a linear operator T on a finite dimensional vector space V .

Or

Let P be the operator on \mathbb{R}^2 which projects each vector onto the x -axis, parallel to the y -axis, $P(x, y) = (x, 0)$. Show that P is linear. What is the minimal polynomial for P ?

GROUP—B

(Vector)

(Marks : 40)

5. Answer the following :

1×3=3

(a) Find the constant p such that the vectors

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k}, \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}, \vec{c} = 3\vec{i} + p\vec{j} + 5\vec{k}$$

are coplanar.

(b) Prove that

$$(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) \cdot \vec{d} = (\vec{a} \cdot \vec{d})[\vec{a} \vec{b} \vec{c}]$$

(c) If \vec{a} and \vec{b} lie in a plane normal to the plane containing \vec{c} and \vec{d} , show that

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 0$$

6. If $\vec{a}_1, \vec{b}_1, \vec{c}_1$ and $\vec{a}_2, \vec{b}_2, \vec{c}_2$ are reciprocal system of vectors, prove that

$$\vec{a}_1 \times \vec{a}_2 + \vec{b}_1 \times \vec{b}_2 + \vec{c}_1 \times \vec{c}_2 = \vec{0} \quad 2$$

7. Answer the following questions : $5 \times 3 = 15$

- (a) If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C, prove that

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

is a vector perpendicular to the plane ABC.

- (b) (i) If $\frac{d\vec{p}}{dt} = \vec{u} \times \vec{p}$ and $\frac{d\vec{q}}{dt} = \vec{u} \times \vec{q}$

show that

$$\frac{d}{dt} (\vec{p} \times \vec{q}) = \vec{u} \times (\vec{p} \times \vec{q})$$

- (ii) If $\vec{w} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$, evaluate
- $$\frac{d}{dt} (\vec{w} \cdot \vec{w})$$

3+2=5

- (c) Prove that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

8. Answer the following questions : $10 \times 2 = 20$

- (a) If \vec{c} is a constant vector, prove that

$$\text{div} (r^n (\vec{c} \times \vec{r})) = 0,$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Or

Prove that the necessary and sufficient condition for a vector $\vec{v}(t)$ to have a constant direction is

$$\vec{v} \times \frac{d\vec{v}}{dt} = \vec{0}$$

- (b) Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x, y, z) = yz\vec{i} - xz\vec{k}$ and C is the line segment from $(-1, 2, 0)$ and $(3, 0, 1)$.

Or

Find the work done when a force

$$\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$$

moves a particle in xy -plane from $(0, 0)$ to $(1, 1)$ along the parabola $y^2 = x$.
