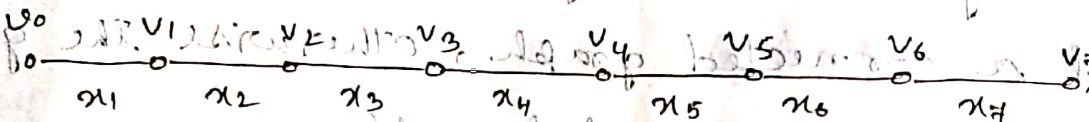
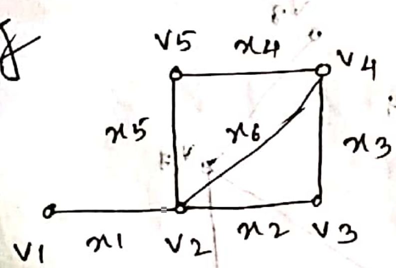
Walks and Connectedness:

A walk of a graph G is an alternating sequence of pts and lines $v_0, x_1, v_1, x_2, \ldots, v_{n-1}, x_n, v_n$ beginning and ending with pts in which each line is incident with the two points immediately preceeding and following it



eg



$v_1, x_1, v_2, x_5, \cancel{x_5}, x_5, \cancel{v}\ v_2, x_2, v_3$

If $v_0 = v_n$, then we have a closed walk.

eg In the above figure,

$v_1, x_1, v_2, x_5, v_5, x_5, v_2, x_2, v_3, x_2, v_2, x_1, v_1$.

A walk is a trail if all the lines are distinct and a path if all the pts are distinct (and thus necessarily all lines). eg in the above fig

(18)

$v_1, x_1, v_2, x_6, v_4, x_3, v_3, x_2, v_2, x_5, v_5$ — trail.

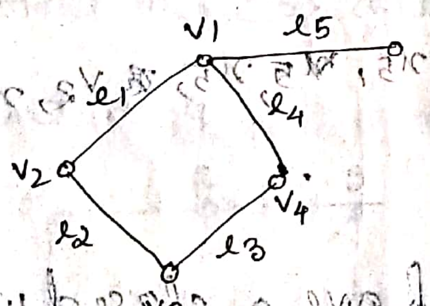
$v_1, x_1, v_2, x_5, v_5, x_4, v_4, x_3, v_3$ — Path.

A closed path is called a cycle (C_n)

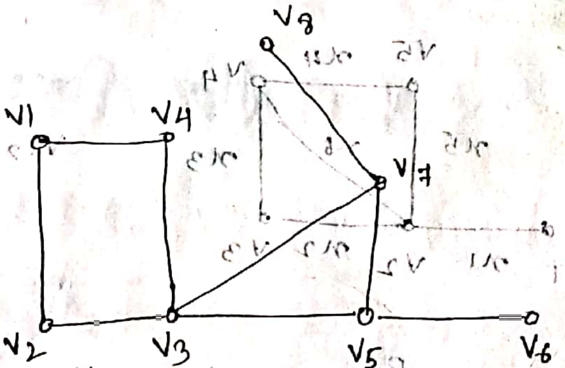
A closed path with 3 pts is called a triangle.

Let G be a graph. If there is a path in G from any vertex to any vertex of G, then G is called a connected graph; otherwise the graph is called a disconnected graph.

eg



G



G

Connectedness is an equivalence relation on the set V

(i) $u c u$, $u \in V$

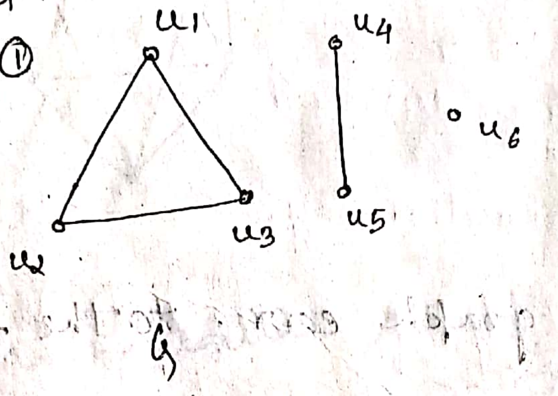
(ii) $u c v \Rightarrow v c u$, $u, v \in V$

(iii) $u c v, v c w \Rightarrow u c w$

A subgraph of a graph G is called a <u>component</u> of G if it is connected and is not properly contained in any connected subgraph of G.

Every graph can be partitioned into finite number of components. so we find that a graph G is connected graph iff G has only one component.

eg. (i)



G has components [u_1], [u_4], [u_6]

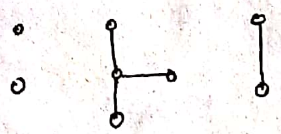
(ii)

→ 4 components

(iii)

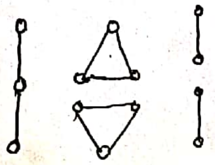
→ 2 components

(iv)



→ 4 components.

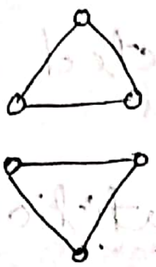
(v)



→ 5 components

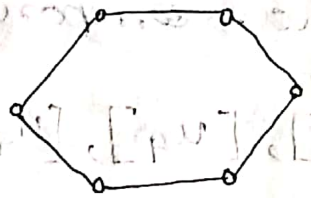
Ex. Construct a dis-connected graph of order 6 and side 6.

Sol^n:



Ex. Construct a ~~dis~~ connected graph of order 6 & side 6.

Sol^n:



C_6

Ex. Does $\exists$ a simple graph corr. to the following degree

(i) 0, 2, 2, 3, 4

$\nexists$ a graph with the given degree as there is ~~are~~ odd number of vertices with odd degree i.e there is only one vertex with odd degree.

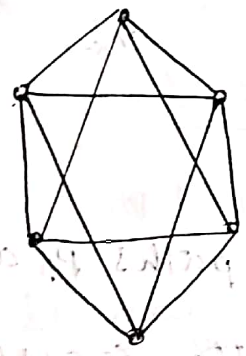
(ii) 1, 1, 2, 3

$\nexists$ a graph with the given degrees as there are 3 vertex with odd degree.

(iii) 2; 3, 3, 4, 5, 6

∄ a graph with ~~a vertex~~ given degree as there are 3 vertex with odd degree.

Ex. Does ∃ a 4-regular graph with 6 pts ?; yes, there exists.



Ex. Does ∃ a 5-regular graph with 11 pts ?

If possible, let ∃ a 5-regular graph G with 11 points.

then, the sum of the degrees of all vertices of the graph $= 5 \times 11 = 55$, an odd integer.

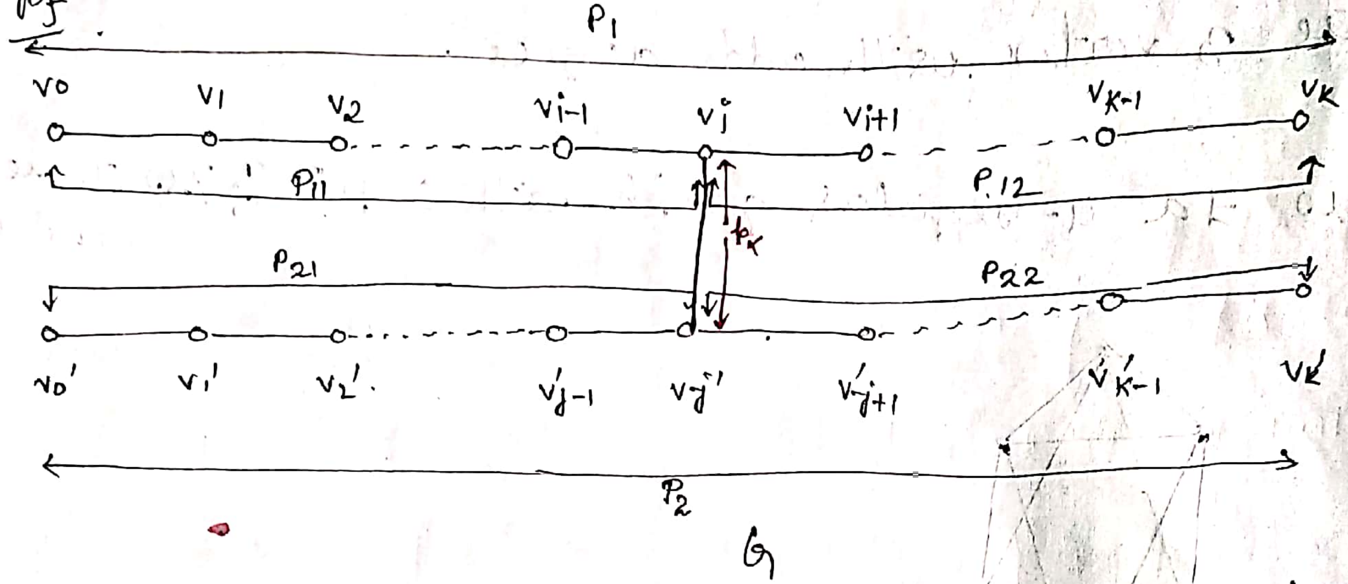
which is a contradiction.

Hence, ∄ a 5-regular graphs with 11 pts.

6/08/10
Friday

Ex. In a connected graph, any two longest path have a common point.

Pf :



Consider any two longest paths P_1 and P_2 of a graph G. Let P_1 be denoted by the sequence $v_0, v_1, v_2, \dots, v_k$ and P_2 by the sequence $v_0', v_1', \dots, v_k'$.

Assume that P_1 and P_2 have no common pt.

Since the graph G is connected then for some i, $0 \le i \le k$ and for some j, $0 \le j \le k$, $\exists$ a $v_i - v_j'$ path P_x such that all points of P_x other than v_i and v_j' are different from those of P_1 and P_2.

Let t_1 = length of $v_0 - v_i$ path P_{11}

t_2 = length of $v_i - v_k$ path P_{12}

& t_1' = length of $v_0' - v_j'$ path P_{21}

t_2' = length of $v_j' - v_k'$ path P_{22}

l_α be the length of path P_α.

we have, $l_1 + l_2 = l_1' + l_2' =$ length of the longest path in graph G.

and also $l_\alpha > 0$

Let $l_1 \geq l_2$

$l_1 \geq l_2$ so that

$l_1 + l_1' \geq l_1 + l_2 = l_1' + l_2'$

Now, it may be verified that the paths P_{11}, P_α, and P_{21} togethere constitute a $v_0 - v_0'$ path with length equal to $l_1 + l_\alpha + l_1' = (l_1 + l_1') + l_\alpha$

$\geq (l_1 + l_2) + l_\alpha > l_1 + l_2$,

a contreadiction.

$\therefore$ In a connected graph, any two longest path have a common point.

Thm: Let G be a connected graph with $p \geq 3$ pts. Then $\omega(G) = q$ iff G has no triangles.

Pf: Let $G(p, q)$ be triangle free graph with $p \geq 3$ pts.

we know that $\omega(G) \leq q \rightarrow (1)$

By defn of the intersection number, G is isomorphic to an intersection graph $\Omega(F)$ on a set S with $|S| = \omega(G)$. For each point v_i of G, let S_i be the corresponding set. Because G has no triangles, no elt. of S can belong to more than two of the sets S_i (otherewise those sets will be mutually adjacent & thus forming a triangle in $\Omega(F) \cong G$) and $S_i \cap S_j \neq \phi$ iff $v_i - v_j$ is a line of G. Thus we can form a 1-1 correspondence between the lines of G and those elements of S which belong to exactly two sets S_i.

so, $|S| \geq q$

$\therefore \omega(G) \geq q \rightarrow (2)$ $\therefore (1), (2) \Rightarrow \omega(G) = q$.

Conversely, let $\omega(G) = q$. We are to S/T G don't have any triangle.

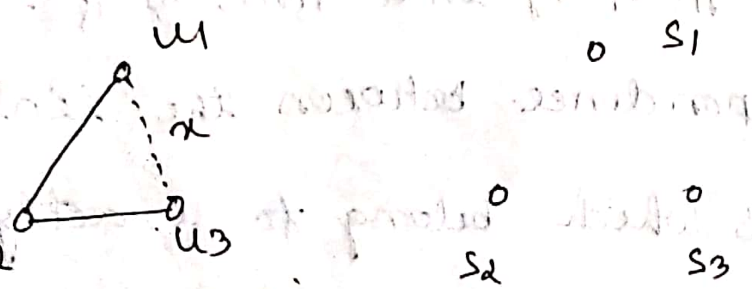
Assume that G has a triangle. Let G_1 be maximal triangle free spanning subgraph of G.

Now, we have $\omega(G_1) = q_1 = |X(G_1)|$.

Suppose that $G_1 \cong \Omega(F)$, where F is a family of subsets of some set S with cardinality q_1 i.e $|S| = q_1$.

Let x be a line of G not in G_1 and consider $G_2 = G_1 + x$.

Since G_1 is maximal triangle free, G_2 must have a triangle say u_1, u_2, u_3 where $x = u_1u_3$.



Denote by S_1, S_2, S_3 the subsets of S corr. to u_1, u_2, u_3. Now if u_2 is adjacent to only u_1

and u_3 in G_1 replace S_2 by a singleton chosen from $S_1 \cap S_2$ and add that elt to S_0 or otherwise replace S_3 by the union of S_3 and any elt in $S_1 \cap S_2$. In either case this gives a family F' of distinct subsets of S such that $G_2 \cong \Omega(F')$.

Thus $\omega(G_2) \leq q_1$ while $|X(G_2)| = q_1 + 1$.

If $G_2 \cong G_1$, there is nothing to prove. But if $G_2 \neq G_1$, then let ~~let $\omega(G_2) \leq q_2 = |X(G_2)|$~~

~~let~~ $|X(G)| - |X(G_2)| = q_0$

$\therefore G$ is an intersection graph on a set with $q_1 + q_0$ elts.

However, $q_1 + q_0 = q - 1$

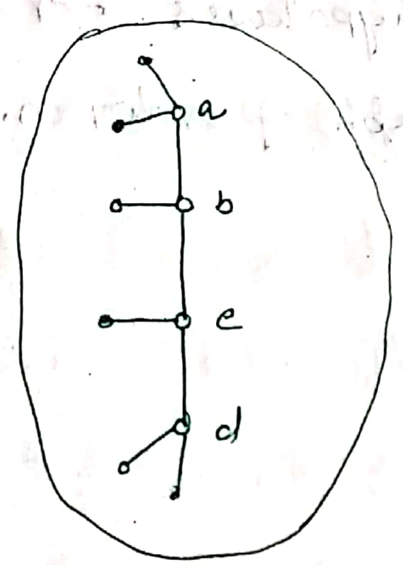
$q_1 + q_0 = q - 1$
$\Rightarrow q_1 + q_0 < q$

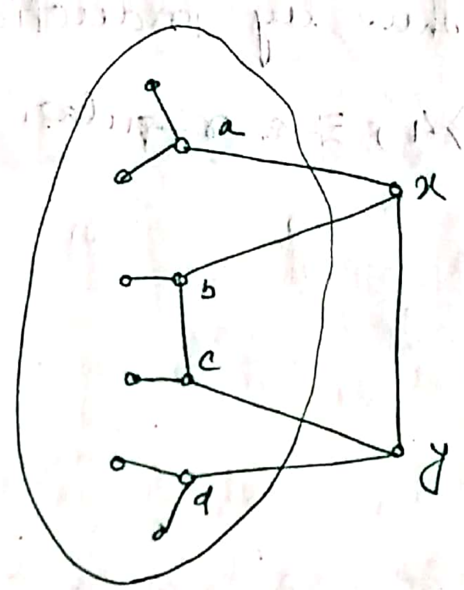
Thus, $\omega(G) < q$, a contradiction.

$\therefore G$ must be triangle free.

Ex: P/T for every even integer $n \geq 4$, $\exists$ a 3-regular graph of order n.

Soln:





fig(a) G $\qquad$ G' fig(b)

We shall use induction on n. If $n = 4$, Then

K_4:  is the only 3-regular graph of order 4

We assume that the result for lesser values of even n $(n > 4)$: Consider a cubic graph G of order $n-2$ such that $ab, bc, cd \in X(G)$ & consider a graph K_2. Let the pts of K_2 be x & y. Remove the lines ab & cd from G. and add lines ax, bx, cy, dy.

The graph G' thus obtain in a 3-regular graph on n-pts.

Thus by induction hypothesis, for every even integer $n \geq 4$, $\exists$ a regular graph of order n.