Th^m Let G be a connected graph with at least 58(1)
three points. The following statements are equivalent:

(1) G is a block. $\left.\begin{matrix} ① \Rightarrow ② \\ ① \Rightarrow ③ \end{matrix}\right\}$ '14

(2) Every two points of G lie on a common cycle.

(3) Every point and line of G lie on a common cycle.

(4) Every two lines of G lie on a common cycle.

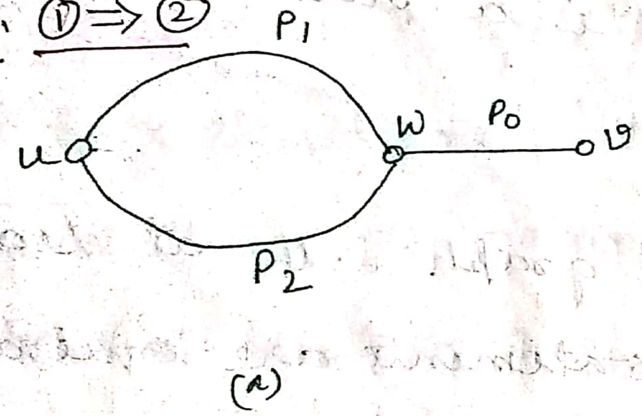
(5) Given two points and one line of G, there is a path joining the points which contains the line.

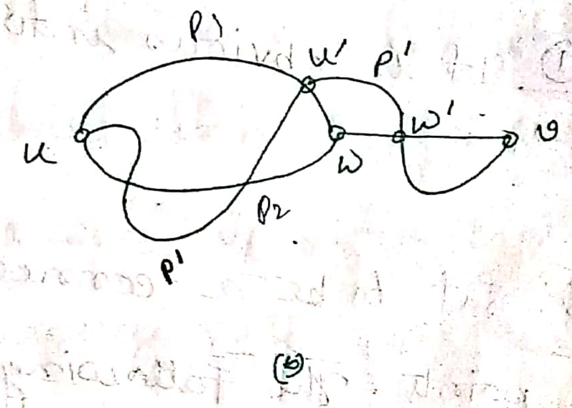
(6) For every three distinct points of G, there is a path joining any two of them which ~~complains~~ contains the third.

(7) For every ~~pair~~ three points of G, there is a path joining any two them which does not contain the third.

Pf: Copy - P - 47 - 53

Pf: ① ⇒ ②





Let u and v be distinct points of G and let U be the set of points different from u which lie on a cycle containing u. Since G has at least three points, no cut points & has no bridge, therefore

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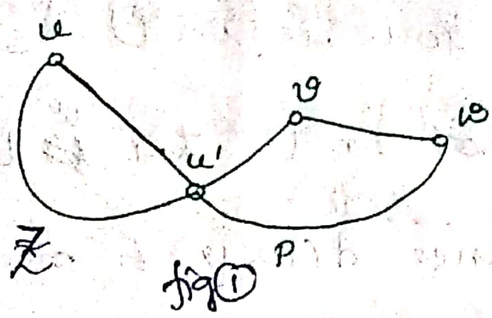
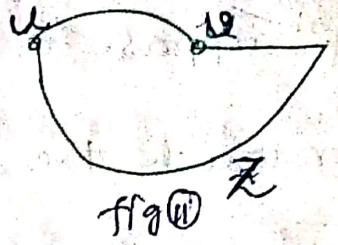
every point adjacent to u is in U. So U is not empty. Suppose v is not in U. Let w be a point in U for which the distance $d(w, v)$ is minm. Let P_0 be the shortest w–v path and let P_1 and P_2 be the two u–w paths of the cycle containing u and w. Since w is not a cut-point, there is a u–v path P' (which is also in P_0) not containing w. Let w' be the point nearest to u in P', which is also in P_0. Let u' be the last pt. of the u–w' subpath of P' into either P_1 or P_2. WLOG, we assume u' in P_1.

Let Q_1 be the u–w' path consisting of the u–u' subpath of P_1 and u'–w' subpath of P'. Let Q_2 be the u–w' path consisting of P_2 followed by the w–w' subpath of P_0. Then Q_1 & Q_2 are disjoint u–w' paths and together they form a cycle, so w' is in U. Since w' is on a shortest w–v path, therefore $d(w', v) < d(w, v)$. This contradicts our choice of w ~~provid~~ proving that u and v must lie on a common cycle.

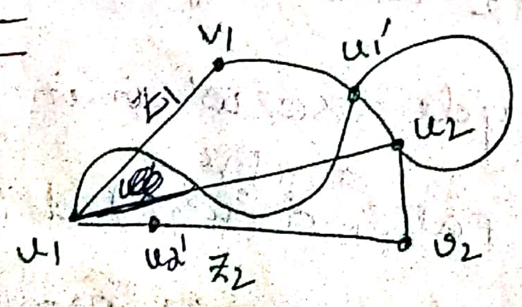
② ⇒ ③





Let u be a point and vw a line of G. Let Z be a cycle containing u and v. A cycle Z' containing u and vw can be formed as follows: If w is on Z, then Z' (fig ⑪) consists of vw together with the v-w path of Z containing u. If w is not on Z, there is a w-u path P not containing v, (fig ①) since otherwise v would be a cutpoint. Let u' be the 1st point of P in Z. Then Z' consists of vw followed by the w-u' subpath of P and u'-v path in Z containing u.

③ ⇒ ④ -

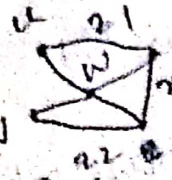


Considere the line u_1v_1 & u_2v_2 of G. Let Z_1 denote the cycle containing u_1v_1 and u_2. Let Z_2 denote the cycle containing u_1 & u_2v_2. If v_2 lies on Z_1. or v_1 lies on Z_2, there is nothing to prove. Suppose, v_2 does not lie on Z_1 and v_1 does not lie on Z_2.

Let, u_1' be the nearest point to v_1 on Z_2 & v_1-u_2 path of Z_1. Let u_2' be the nearest point to v_2 which lie on Z_1 and u_1-v_2 path of Z_2

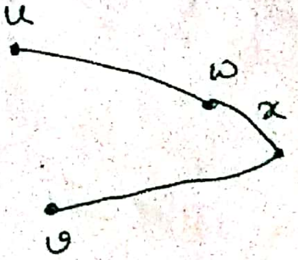
Let, us construct a cycle consisting of u_2-u_1' path of Z_2 followed by $u_1'-v_1$ path of Z_1 followed by v_1u_1 followed by u_1-u_2' path of Z_1 followed by $u_2'-v_2$. path of Z_2 then followed by the line v_2u_2. This is the reqd cycle.

④ ⇒ ⑤ Any two points of G are incident with one line each, which lie on a cycle by ④. Hence any two points of G lie on a cycle, and we have ②, so also ③.



Let u and v be distinct points and x be a line of G. By statement ③, there are cycles Z_1 containing u and x, and Z_2 containing v & x. If v, is on Z_1 or u is on Z_2, there is clearly a path joining u and v containing x. Thus, we need only consider the case where v is not on Z_1 & u is not on Z_2. Begin with u and proceed along Z_1 until reaching the first point w of Z_2, then take the path on Z_2 joining w and v which contains x. This walk constitutes a path joining u and v that contains x.

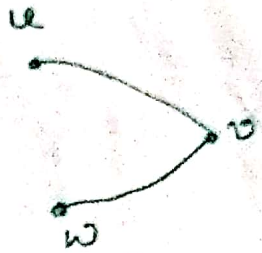
⑤ ⇒ ⑥ Let u, v, w be three distinct points of G. Let x be any line incident with w. By ⑤, there is a path joining u and v which contains x and hence must contain w.



∴ ⑤ ⇒ ⑥

⑥ ⇒ ⑦

Let u, v, w be 3-distinct points of G.



By ⑥, there is a $u-w$ path P containing v. Then $u-v$ subpath of P does not contain w.

$\therefore$ ⑥ $\Rightarrow$ ⑦

⑦ ⇒ ① For any two points u and v of G, no point lies on every $u-v$ path. Hence G is a block.

$\therefore$ ⑦ $\Rightarrow$ ①.

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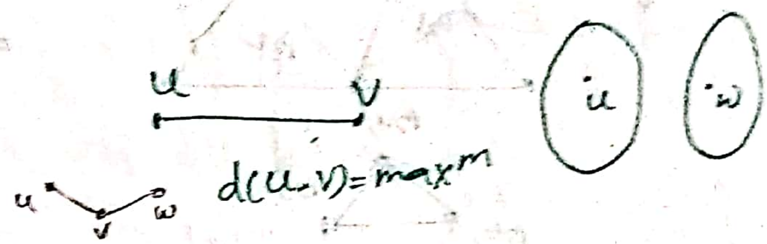
Th^m 4*: Every non-trivial connected graph has at least two points which are not cut points.

Pf: Let u, v be a points of a graph G such that $d(u,v)$ is maximum. Assume that v is a cut pt. Then there is a point w in a different component of $G-v$ than u. Hence v is in every path joining u and w, so $d(u,w) > d(u,v)$, which is impossible. Therefore v and similarly u are not cut pts of G.



u & v are end pts of G.