Unit 3

Block, cut points, bridges, Block graphs, Cut point graphs, Trees, characterization of trees.

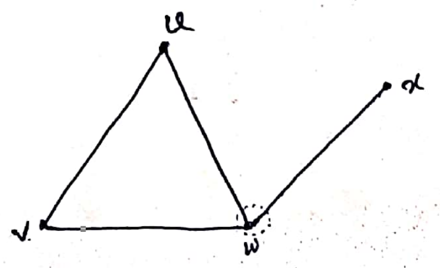
cut point, bridge and Block:

Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components. A cut vertex or cut point of a connected graph G is a vertex whose removal increase the number of components. Clearly if v is a cut vertex of a connected graph G, G - v is disconnected.

Analogously, an edge whose removal produces a graph with more connected components ~~than the original graph~~ is called a cut edge or bridge.

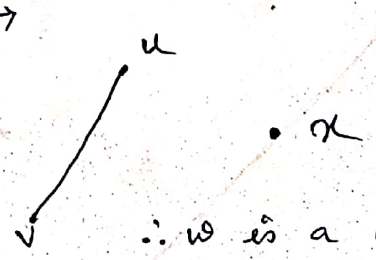
A non-separable graph is connected, non-trivial and has no cut-point.

eg.

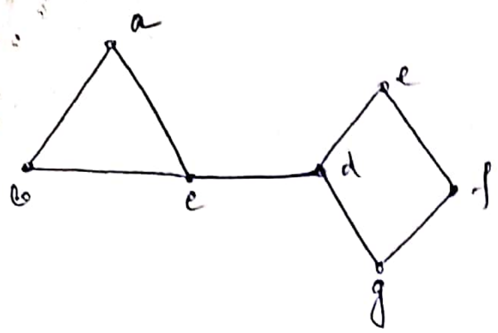


G_1

if we remove w then G_1 will have 2 components:



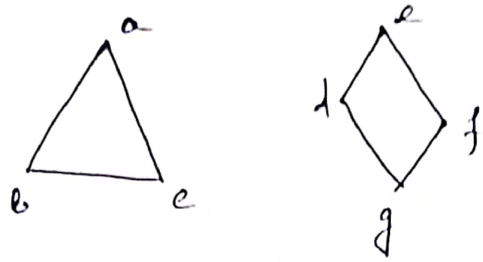
∴ w is a cut-point



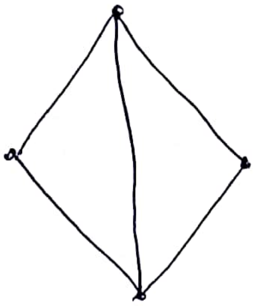
G_2

→

if we remove cd then G_2 has 2 components:

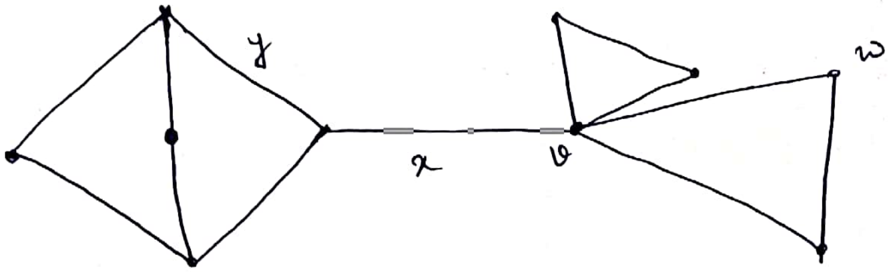


∴ cd is a bridge.

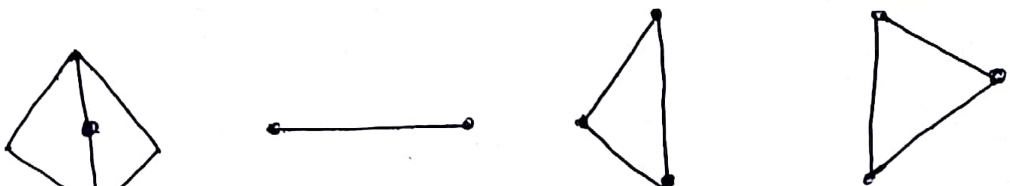


→ non-separable graph.

A <u>block</u> of a graph is a maximum non-separable subgraph.



Here v is a cut point, x is a bridge. But w is not a cut point & y is not a bridge. The blocks are:



Thm: Let v be a point of a connected graph G. The following statements are equivalent:

(1) v is a cut point of G. ① ⇒ ② (17)

(2) There exists points u and w distinct from v such that v is on every $u-w$ path.

(3) There exists a partition of the set of points $V-\{v\}$ into subsets U and W such that for any points $u \in U$ and $w \in W$, the point v is on every $u-w$ path.

Proof: ① ⇒ ③

Let v is a cut point of G.

Since v is a cut point, $G-v$ is disconnected and has at least two components. Form a partition of points $V-\{v\}$ by considering two subsets U and W such that U consists of the points of one component & W the points of the other. Then the two points $u \in U$ and $w \in W$ lie in different components of $G-v$. Therefore every $u-w$ path in G must contain the point v.

③ ⇒ ② ② is a particular case of ③. Here U may be constructed as the set of points which are connected to $u \in G-v$ and W as the set of points which are connected to $w \in G - v$.

(a) $\Rightarrow$ (1) If v is on every path in G joining u and w then there can't be a path joining these points in $G - v$. Thus $G - v$ is disconnected, so v is a cutpoint of G.

Thm: Let x be a line of connected graph G. The following statements are equivalent:

(I) x is bridge of G.

(II) x is not on any cycle of G.

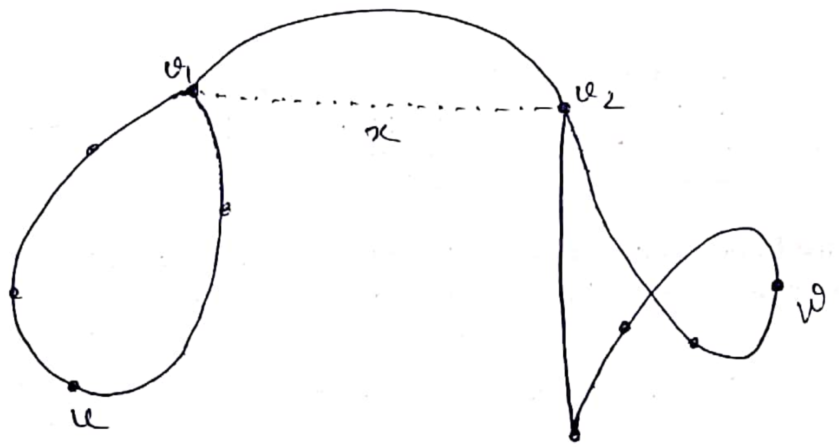
(III)(B) $\exists$ points u and w of G such that the line x is on every path joining u and w.

(IV) $\exists$ a partition of V into subsets U and W such that ~~the line x is~~ for any points $u \in U$ and $w \in W$, the line x is on every path joining u and w.

Proof: (III) $\Rightarrow$ (II):

Let v_1, v_2 denote the end points of the line x. Then v_1 and v_2 must be two points on every path joining u and w. If x lies on a cycle then v_1 and v_2 are connected in $G - x$ and therefore u and w are connected in $G - x$. This contradicts the fact

that x lies on every path joining u and w in G.
so x is not on any cycle of G.



(ii) $\Rightarrow$ (i) Let v_1 and v_2 be the end points of the line x. Since x is not on ~~every~~ any cycle of G, then v_1 and v_2 are connected by x only. Then in $G-x$, v_1 and v_2 are not connected and hence they lie in two components of $G-x$. Therefore x is a bridge of G.

(i) $\Rightarrow$ (iv) Since x is a bridge of G, $G-x$ is disconnected and has at least two components. Form a partition of V by letting U consist of the points of one component and W, the points of other component. The any two points $u \in U$ & $w \in W$ lie in different components of $G-x$. Therefore every $u-w$ path in G contain the line x.

(iv) $\to$ (i): It is obvious that (iii) is a particular case of (iv).

Thm Let G be a connected graph with at least three points. The following statements are equivalent:

① G is a block. ① ⇒ ② , ① ⇒ ③ } 14

(2) Every two points of G lie on a common cycle.

(3) Every point and line of G lie on a common cycle.

(4) Every two lines of G lie on a common cycle.

(5) Given two points and one line of G, there is a path joining the points which contains the line.

(6) For every three distinct points of G, there is a path joining any two of them which ~~contains~~ contains the third.

(7) For every ~~poi~~ three points of G, there is a path joining any two them which does not contain the third.

Pf: Copy - P - 47 - 53

Thm Every non-trivial connected graph has at least two points which are not cut points.

Pf: Copy P-55

Scanned with CamScanner

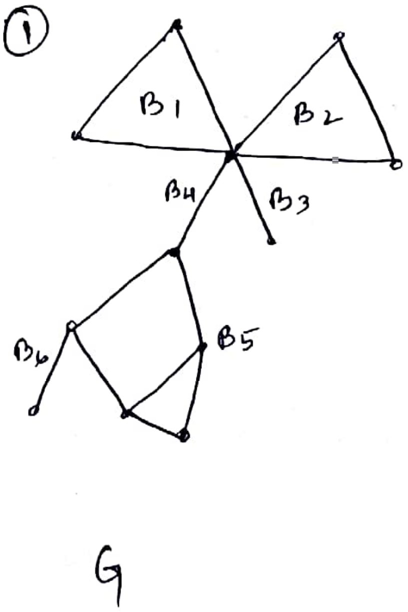
(iv) ⇒ (i) It is obvious that (iii) is a particular case of (iv).

Block graphs:

Let the blocks of the graph G is denoted by $B_1, B_2, \dots, B_k$. Let S be the set of all points and lines of G. Let S_i be the set of points and lines of B_i and $F = \{S_1, S_2, \dots, S_k\}$. Then the intersection graph $\Omega(F)$ is called the Block graph of G and is denoted by $B(G)$.

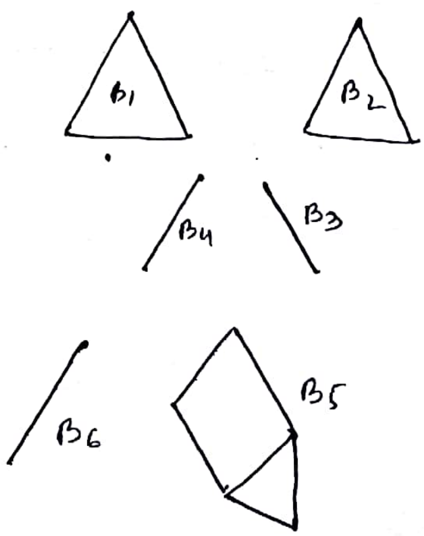
Remark: Every block of G corresponds to a point of $B(G)$ and such two points in $B(G)$ are adjacent whenever the corresponding blocks contain a common cut point of G.

eg ①

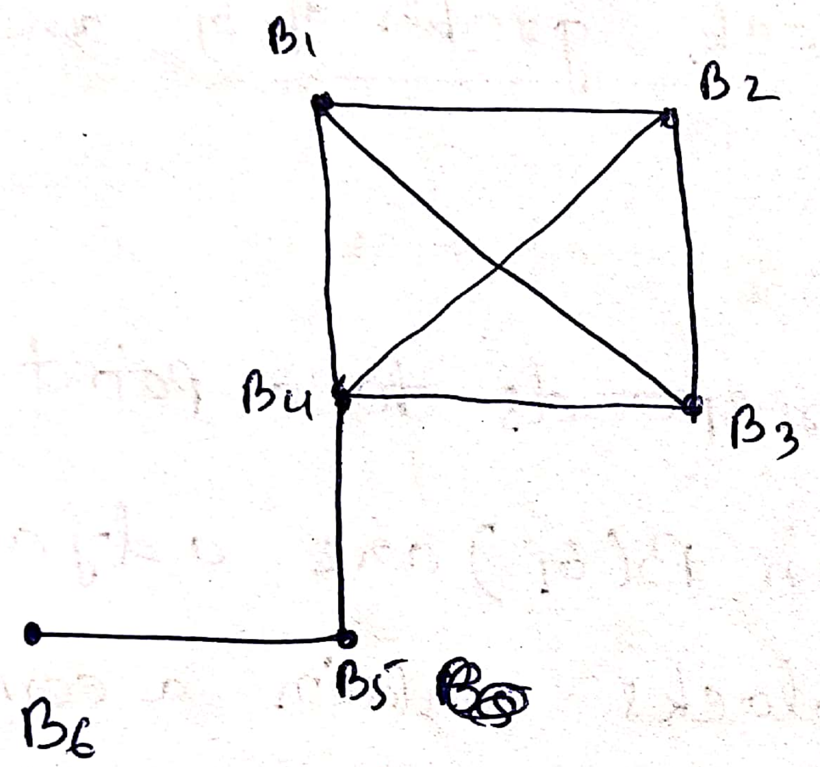


→

Blocks of G are



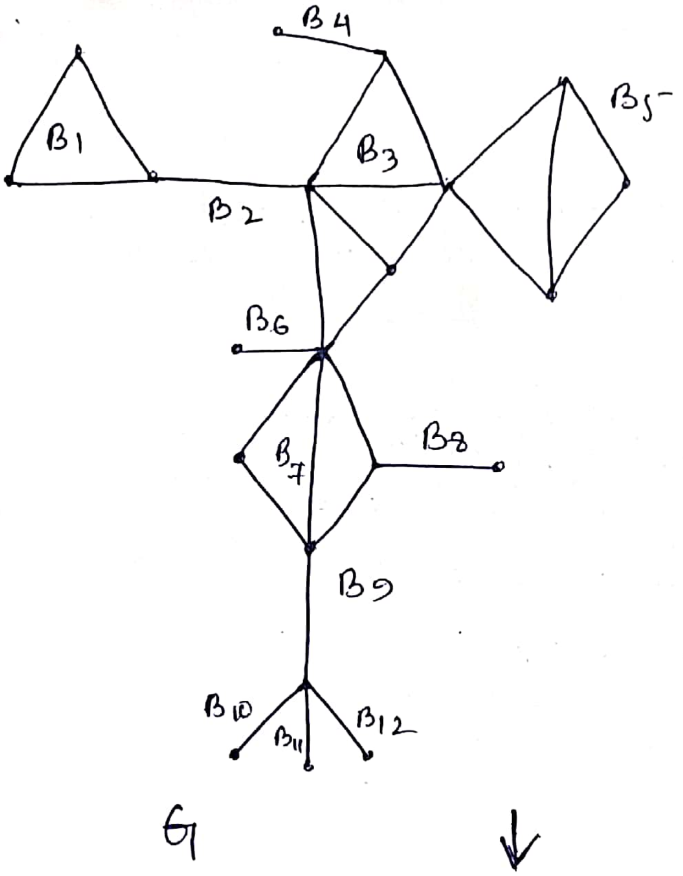
P.T.O→



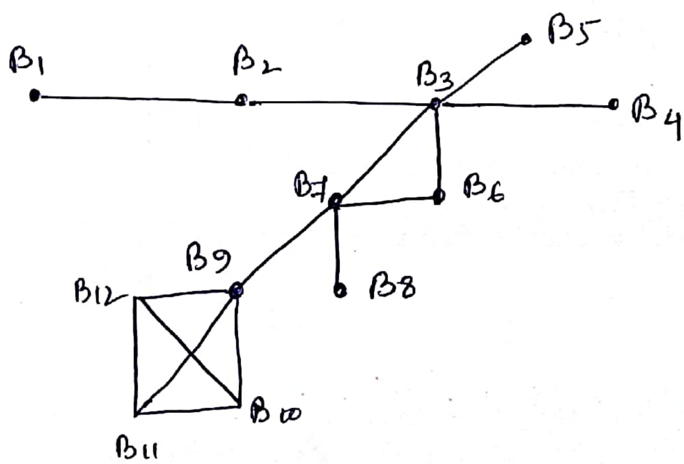
B(G) → the block graph of G.

II

② 7



↓



B(G).

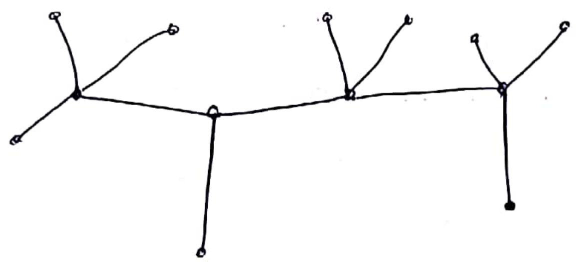
Thm: A graph H is the block graph of some graph if and only if every block of H is complete.

Pf: ~~Rop g~~ Copy – P-58

Trees: A graph is acyclic if it has no cycles.

A tree is a connected graph which does not contain any cycle. It follows from the definition that a tree must be a simple graph, because a loop or parallel edges form a cycle. Any graph without cycles is a forest, thus the components of a forest are trees.

eg.



'15

Thm 1: A graph G is a tree iff there is a unique path between every pair of vertices of G.

Pf: Let G be a tree. Since G is connected, there must exist at least one path between every pair of vertices v_i, v_j in G.

If possible, let there be two distinct paths between v_i and v_j. The union of these paths will form a cycle in G which is a contradiction, since G does not contain any cycle. Hence every pair of vertices in a tree is joined by one and only one path.

Conversely, let there is a unique path between every pair of vertices of G.

G must be connected, since there is a path between every pair of vertices v_i and v_j' in G. If possible, let there be a cycle in G. Then there is at least one pair of vertices v_i and v_j' such that there are two distinct paths between v_i and v_j'. It is a contradiction to the fact that every pair of vertices in G is joined by only one path. Hence G does not contain any cycle and so G is a tree.

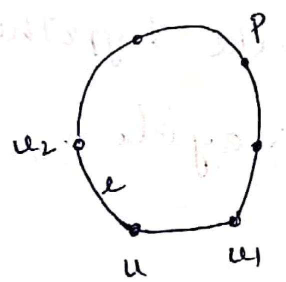
17 **Thm**: Let T be a tree with more than one vertex. Then T contains at least two vertices of degree 1.

Pf: Let T be a tree and v_i, v_j be two vertices of T. Then there exists a unique path from v_i to v_j'. Let S be the set of all paths in T.

Since the number of vertices in T is finite, the number of paths in T is finite, i.e. S is finite. Hence we can find a path P in T with maximal number of vertices. Let P be from the vertex u to the vertex v. We first show that $d(u) = 1$.

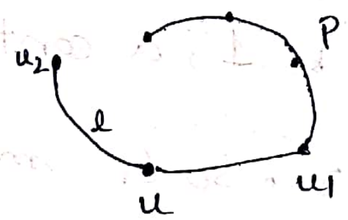
If possible, let $d(u) > 1$. Then u has more than one adjacent vertices. One of these adjacent vertices, say u_1, must be on P. Now, let u_2 be another adjacent vertex of u. Then u_2 can not be on P. For, in that case, starting from u we may go to u_1 and then to u_2 along the path P and come back to u along the other edge, say e, incident with u & u_2, thus forming a cycle (fig ① above). Since u_2 is not a member of P, the path (u_2, e, P) contains more vertices than that of P which contradicts the definition of P. Hence $d(u) = 1$.

Similarly, we can show $d(v) = 1$. Thus the proof is complete.



(contradiction)

fig ①



Path (u_2, e, P) > Path P, which violates the def^n of P.

Thm: If in a graph, $\delta(G) \geq 2$, then G contains a cycle.

Pf: If G does not contain a cycle, then G is a forest. Each component of G is a tree.

But G has at least two end points ($\because \delta(G) \geq 2$ i.e G is a nontrivial tree). So the degree of each end

1993.

points will be 1, a contradiction by the hypothesis that $\delta(G) \geq 2$. So G must contain a cycle. //

Scanned with CamScanner

Thm: The following statements are equivalent for a graph G.

(1) G is a tree.

(2) Every 2 points of G are joined by a unique path

(3) G is connected and $p = q + 1$

(4) G is acyclic & $p = q + 1$

(5) G is acyclic and if two non-adjacent points of G are joined by a line x, the $x + G$ has exactly one cycle.

(6) G is connected, its not K_p for $p \geq 3$, and if any two nonadjacent points of G are joined by a line x, then $G + x$ has exactly one cycle.

(7) G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if any two nonadjacent points of G are joined by a line x, then $G + x$ has exactly one cycle.

Pf: (1) $\Leftrightarrow$ (2) $\rightarrow$ Proved in Thm 1.

(2) $\Rightarrow$ (3). Since every 2 points of G are joined by a unique path, so G is connected.

We prove, $p = q + 1$, by induction.

It is true for $p = 1$ $(q = 0)$, $p = 2$, $(q = 1)$.

Assume it is true for all connected graphs with fewer than p points. If G has p points, then the removal of any line x from G disconnects G. So $G-x$ has at least 2 components. Let $n<p$ be the number of points of one component, then the other component of $G-x$ will have $p-n$ points. By induction hypothesis, the no. of lines q_1 of the component having n points is $n-1$ and the no. of lines q_2 of the component having $p-n$ points is $p-n-1$.

The no. of lines q of G is

$$q = q_1 + q_2 + 1$$
$$= (n-1) + (p-n-1) + 1.$$
$$= p-1$$
$$\Rightarrow p = q+1.$$

③ ⇒ ④ : Assume that G has a cycle of length n (say) Then there are n points and n lines on the cycle. For each of the $p-n$ points not on the cycle, there is a distinct line on the shortest path connecting these points to a point of the cycle. Hence any point of G corresponding to a distinct line of G, so $q \geq p$, which is a contradiction to the fact that $p = q + 1$.

$\therefore$ G must be acyclic.

④ ⇒ ⑤ : G is acyclic ⇒ each component of G is a tree.

Let there be k components and let $q_1, q_2, \ldots, q_k$ be the no. of lines in the respective components.

Let $p_1, p_2, \ldots, p_k$ be the no. of ~~lines~~ points in the respective components.

In each component,

$$p_i = q_i + 1$$

$$\therefore p = \sum p_i = \sum (q_i + 1)$$

$$= \sum q_i + k = q + k.$$

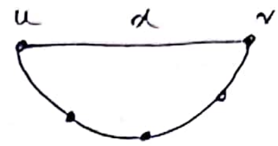
But in G we have; $p = q + 1$

$\therefore k = 1$ (ie G only one component ~~that~~ which is G itself)

$\therefore$ G is connected.

So G is connected and acyclic, hence G is a tree

There is exactly one path connecting any 2 points of G. Let the line x joints 2 non-adjacent points u and v of G. Then the single path connecting u and v together with the line x forms exactly one cycle in G.



⑤ ⇒ ⑥

Given that by joining two non-adjacent points u and v by a line x we get exactly one cycle in $G+x$. So the two non-adjacent points u and v must be connected by a path in G other than x. Hence G is connected.

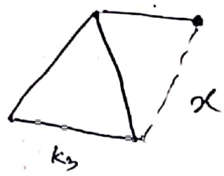
If G is K_p for $p \geq 3$, then G must contain a cycle. But G is acyclic, so G is not $K_p, p \geq 3$.

⑥ ⇒ ⑦ We shall prove that any two points of G are joined by a unique path. Since by joining a non-adjacent points u & v by a line x, we get a cycle, so u & v must be connected by a path other than x. Hence G is connected.

If the points u & v are connected by 2 paths in G, then G has a cycle. If a cycle contains 4 or more points then we get 2 cycles out of it by joining a line

x to it. This is not the case by hypothesis. Hence the cycle of G is K_3 which must be a proper subgraph of G, since by hypothesis $G \neq K_p$, $p \geq 3$.

Since G is connected, we may assume that there is another point in G which is joined to a point of this K_3. Then it is clear that ~~a~~ a line



x may be added so as to form at least 2 cycles in $G+x$. This is not the case by hypothesis. Therefore K_3 can't be a proper subgraph of G and also any two points of G are connected by a unique path. Then this implies that G is connected & $p = q+1$.

Finally, K_3 is not a proper subgraph

$\Rightarrow G \neq K_3 \cup K_1$ or $K_3 \cup K_2$.

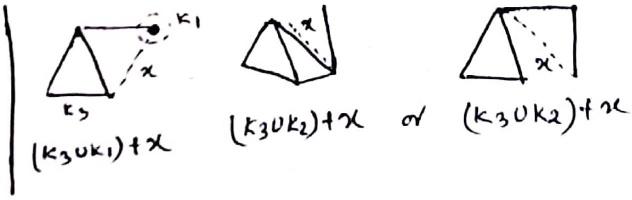
(7) $\Rightarrow$ (1) If G has a cycle, then that cycle must be a triangle which is a component of G by an argument in the preceding case. This component has 3 points and ~~to~~ 3 lines. All other components of G must be trees and in order to make $p = q+1$, there can be only one other component.

If this tree contains a path of length 2, it will be possible to add a line x to G and obtain 2 cycles in $G+x$. Thus this tree must be either K_1 or K_2 and hence G must be either $K_3 \cup K_1$ or $K_3 \cup K_2$, which are the graphs that have been excluded.

(ie $G \neq K_3 \cup K_1$, $K_3 \cup K_3$ → by our hypothesis).

∴ our assumption that G has a cycle is wrong, ie G is acyclic. But if G is acyclic & $p = q+1$, then G is connected since ④ ⇒ ⑤ ⇒ ⑥.

∴ G is a tree.



Ex Draw a graph having the given properties or explain why no such graph exists.

(a) A connected acyclic graph with ten points & eight lines.

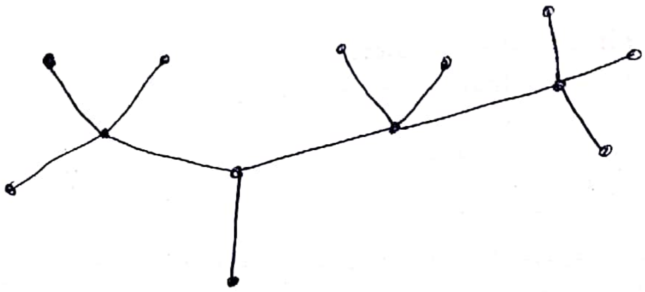
(b) Tree, 5 vertices of degrees 1, 3, 4, 4, 6, 8.

(c) Tree, 13 vertices with 9 vertices of degree 1, 3 vertices of degree 4 & one vertex of degree 3.

Soln (a) Such a graph does not exist. For, the graph is connected and acyclic implies it is a tree and it has 10 points. Hence it must have $10-1 = 9$ lin

(b) Such a graph does not exist. Since the ~~[illegible]~~ graph is connected a tree, it must have at least two vertices of degree 1. But there is given only one vertex of degree 1.

(c)



Ex Does there exists a tree T with 8 vertices such that the sum of the degrees of the vertices is 16? Justify your answer.

Soln: Since T is a tree with 8 ~~vertices~~ points, it must have $8-1=7$ lines. Consequently, the sum of the degrees of the vertices must be $7\times2=14$. Hence such a tree does not exist.

Ex. Prove that a connected graph G with $n\geq2$ vertices is a tree if and only if the sum of the degrees of the vertices is $2(n-1)$.

Proof: Let G be a tree with n vertices. Then it must have $n-1$ edges. Consequently, the sum of the degrees of the vertices is $2(n-1)$.

Scanned with CamScanner

Conversely, let G be a connected graph with n vertices and the sum of the degrees of the vertices be $2(n-1)$. The the no. of edges in G is $\frac{2(n-1)}{2} = n-1$. Thus G is a connected graph with n vertices and $n-1$ edges. Hence G is a tree.

Ex Draw a graph having the given properties or, explain why no such graph exist.

(a) 4 edges, 6 vertices & no cycle.

(b) Tree, all vertices of degree 2.

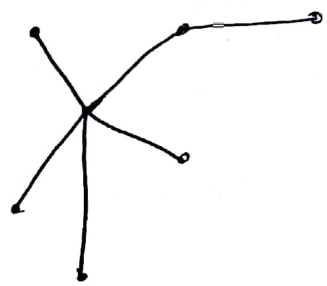
(c) Tree, six vertices having degrees 1, 1, 1, 1, 3, 3.

(d) Tree, five vertices having degrees 1, 2, 2, 3, 4.

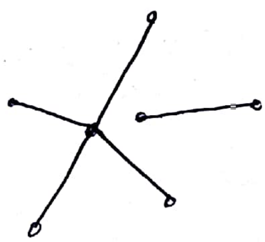
Ex Can we draw a tree with 13 vertices having four vertices of degree 3, three vertices of degree 4, and six vertices of degree 1? Justify your answer.

Ex. Which of the following graphs are trees & which are not? Justify.

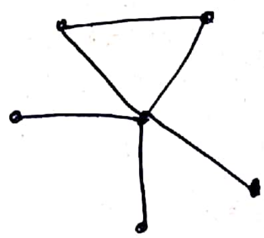
(a)



(b)



(c)



Ex. A certain tree T of order 35 has 25 points of degree 1, 2 points of degree 2, 3 points of degree 4, 1 point of degree 5 and 2 points of degree 6. It also contain 2 points of the same degree x. What is x ?

Ans: For a tree, $p = q + 1$

Here $p = 35 \Rightarrow q = 35 - 1 = 34$

$\therefore 25 \times 1 + 2 \times 2 + 3 \times 4 + 1 \times 5 + 2x = 34 \times 2$

$\Rightarrow 58 + 2x = 68 \Rightarrow x = 5.$

Ex. A tree with 50 end points has an equal number of points of degree 2, 3, 4 & 5 and no point of degree greater than 5. What is the order of T.

Ans: $p = 50 + x + x + x + x = 50 + 4x$

$q = p - 1 = 49 + 4x$

$\therefore 50 \times 1 + 2x + 3x + 4x + 5x = 2(49 + 4x)$

$\Rightarrow 50 + 14x = 98 + 8x$

$\Rightarrow x = 8$

$\therefore p = 50 + 4 \times 8 = 82.$

Ex. Let the average degree of a connected graph G of order p be greater than 2. Prove that G has at least 2 cycles.

Soln: Total no. of points $= p$. (let)

Average degree $= \frac{\sum \deg v_i}{p} > 2$ $\quad \left| \sum \deg v_i = 2q \right.$

$$\Rightarrow \frac{2q}{p} > 2$$

$$\Rightarrow 2q > 2p \Rightarrow q > p$$

$\therefore$ G contains at least one cycle i.e G is not a tree.

Let x be a line of the cycle. Let $G' = G - x$

$\therefore V(G') = p' = p$

$X(G') = q' = q - 1 \Rightarrow q = q' + 1$

Since $p < q \Rightarrow p < q' + 1$

$\Rightarrow p' < q' + 1 \quad (\because p = p')$

$\Rightarrow q' > p' - 1$

$\therefore$ G' is also not a tree and hence contain a cycle. So G contains at least 2 cycles.

Ex. Determine the no. of points and lines in a tree consisting of 2n pendant points, 3n points of degree 2 and n points of degree 3.

Soln: No. of points $p = 2n+3n+n = 6n$

No. of lines $q = \frac{2n \times 1 + 3n \times 2 + n \times 3}{2} = \frac{11n}{2}$

Now, $p = q+1$

$\Rightarrow 6n = \frac{11n}{2} + 1$

$\Rightarrow n = 2$

$\therefore p = 6 \times 2 = 12$

$q = 11$

Ex. The degrees of the points of a certain tree T of order 13 are 1, 2, 5. If T has 3 points of degree 2, how many end points does it have?

Soln: Let there be x end points.

$p = 13, \quad q = p-1 = 12$

$\therefore x \times 1 + 3 \times 2 + (13 - x - 3) \times 5 = 12 \times 2$

$\Rightarrow x + 6 + 50 - 5x = 24$.

$\Rightarrow x = 8$

$\therefore$ It has 8 end points.

Ex: Let T be a tree with 50 lines. The removal of certain line from T yeilds two disjoint trees T_1 & T_2. Given that the number of points in T_1 equals the no. of lines in T_2, determine the no. of points and no. of lines in T_1 & T_2?

Sol^n: Given $q = 50$

$\therefore p = q + 1 = 51$

Given, $p_1 = q_2$ where $p_1, p_2 \rightarrow$ no. of points in T_1 & T_2 & $q_1, q_2 \rightarrow$ no. of lines in T_1 & T_2.

$= p_2 - 1$

$\Rightarrow p_1 - p_2 = -1 \rightarrow (1)$

Again, $p_1 + p_2 = 51 \rightarrow (11)$

$(1) + (11) \Rightarrow 2p_1 = 50 \Rightarrow p_1 = 25$

$\therefore p_2 = 25 + 1 = 26$, $q_1 = p_1 - 1 = 25 - 1 = 24$

$q_2 = p_2 - 1 = 26 - 1 = 25$

$\therefore T_1$ & T_2 are $\underbrace{(25, 24)}_{(p,\ q)}$, $\underbrace{(26, 25)}_{(p,\ q)}$ graphs.

Ex - Show that every forest of order n with k components has size n-k.

Sol^n: Let F be a forest of order n with k components. So each component of F is a tree.

Given, $p = n = \sum_{i=1}^{k} p_i$.

Let, q be the size of F. Since F has k components,

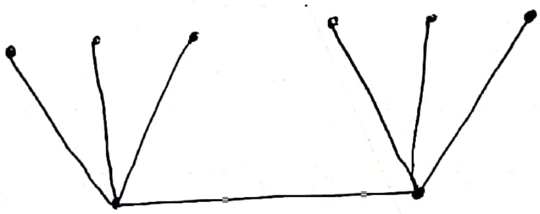
$$\therefore q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1)$$

$$= \sum_{i=1}^{k} p_i - k$$

$$= n - k.$$

Ex Give an example of a tree of order 6 containing 4 points of degree 1 & two points of degree 3.

Ex Give an example of a tree of order 8 containing 6 points of degree 1 & two points of degree 4.



Ex Find all trees T where 75% of the points of T have degree 1 and remaining 25% points has degree 4.

Solⁿ Let n be the no. of points of the tree.

Then, $\frac{75}{100} \times n + \frac{25}{100} \times 4n = 2(n-1)$

$\Rightarrow n = 8$

$p = q + 1$
$\Rightarrow n = q + 1$
$\Rightarrow q = n - 1$

$\frac{75}{100} \times 8 = 6 \rightarrow$ there are 6 pts of degree 1 & 2 pts of degree 4.