Graph Theory

CHAPTER 13

13.1. Introduction

Many situations that occur in computer Science, Physical Science, Communication Science, Economics and many other areas can be analysed by using techniques found in a relatively new area of mathematics called graph theory. The graphs can be used to represent almost any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with relationship among them. In this chapter, we begin with some basic graph terminology and then discuss some important concepts in graph theory with many applications of graphs.

13.2. Basic Terminology

A graph G consists of a set V called the set of **nodes (points, vertices)** of the graph and a set E of edges such that each edge $e \in E$ is associated with ordered or unordered pair of elements of V, that is, there is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V. The set V(G) is called the **vertex set** of G and E(G) is the **edge set**.

The graph G with vertices V and edges E is written as G = (V, E) or G(V,E).

If an edge $e \in E$ is associated with an ordered pair (u, v) or an unordered pair (u, v), where $u, v \in V$, then e is said to **connect** u and v and u and v are called end points of e. An edge is said to be incident with the vertices it joins. Thus, the edge e that joins the nodes u and v is said to be **incident** on each of its end points u and v. Any pair of nodes that is connected by an edge in a graph is called **adjacent nodes**.

In a graph a node that is not adjacent to another node is called an **isolated node**.

A graph G (V,E) is said to be **finite** if it has a finite number of vertices and finite number of edges. (A graph with a finite number of vertices must also have finite number of edges): otherwise, it is a **infinite** graph, if G is a finite, | V (G) | denotes the number of vertices in G and is called the **order** of G. Similarly if G is finite, | E (G)| denotes the number of edges in G and is called the size of G. We shall often refer to a graph of order n and size m an (n, m) graph. If G be a (p, q) graph then G has p vertices and q edges.

Although graphs are frequently stored in a computer as list of vertices and edges, they are pictured as diagrams in the plane in a natural way. Vertex set of graph is represented as a set of points in a plane and edge is represented by a line segment or an arc (not necessarily straight). The objects shown in Fig. 13.1 are graphs.

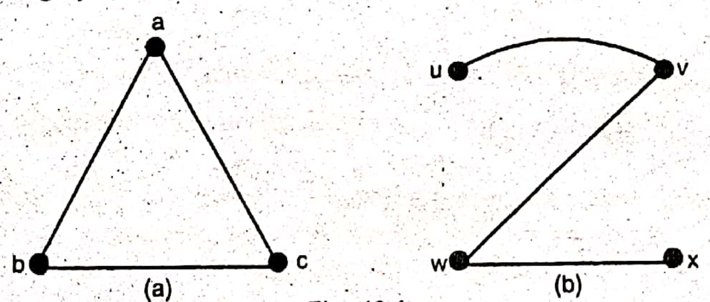


Fig. 13.1

It helps when discussing a graph to label each vertex, often with lower case letters as shown above. In Fig. 13.1 (*a*), V = {*a*, *b*, *c*,} and E = {(*a*, *b*), (*a*, *c*), (*b*, *c*)} the member of vertices and edges are |V(G)| = 3 and |E(G)| = 3 in this graph, the vertices *a* and *b*, *a* and *c* and *b* are *c* are adjacent vertices.

In Fig. 13.1 (*b*), $V = \{u, v, w, x\}$ and $E = \{(u, v), (v, w), (w, x)\}$

Here vertices *u* and v, *v* and *w*, *w* and *x* are adjacent, whereas *u* and *w*, *u* and *x* and *v* and *x* are non adjacent. The number of vertices and edges are |V(G)| = 4 and |E(G)| = 3.

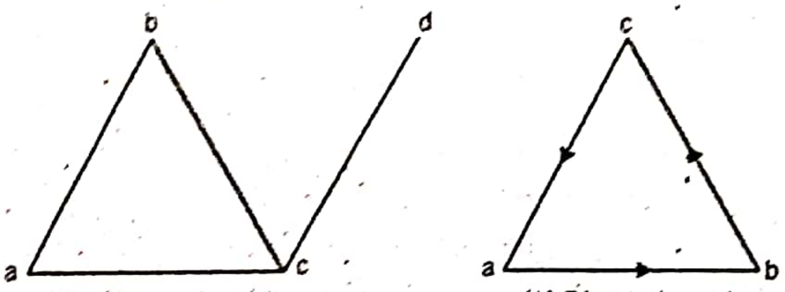
The definition of a graph contains no reference to the length or the shape and the positioning of the edge or arc joining any pair of nodes, nor does it prescribe any ordering of positions of the nodes. Therefore, for a given path, there is no unique diagram that represents the graph, and it can happen that two diagrams that look entirely different from one another may represent the same graph. It is to be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short, what is important is the incidence between edges and vertices are the same in both cases.

Undirected and Directed Graph

An **undirected** graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

Fig. 13.2 (*a*) is an example of an undirected graph we can refer to an edge joining the vertex pair *i* and *j* as either (*i*, *j*) or (*j*, *i*).

A **directed** graph (or **digraph**) *G* consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called directed graph. In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point *u* of *e* to the terminal point *v*. Fig. 13.2 (*b*) is an example of a directed graph.



(*a*) Undirected graph (*b*) Directed graph

Fig. 13.2

Suppose $e = (u,v)$ is a directed edge in a digraph, then

(*i*) *u* is called the **initial vertex** of *e* and v is the **terminal vertex** of *e*

(*ii*) *e* is said to be **incident from** *u* and to be **incident** to v.

(*iii*) *u* is adjacent to v, and v is **adjacent from** *u*

In specifying any edge of a digraph by its end-points, the edge is understood to be directed from the first vertex towards the second.

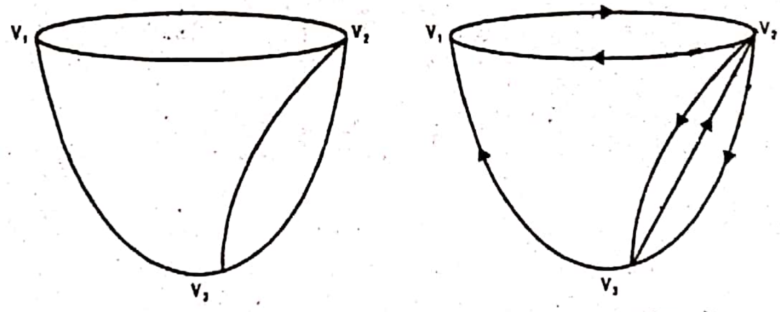
13.3. Simple Graph, Multigraph and Psuedo Graph

An edge of a graph that joins a node to itself is called a **loop** or **self loop** *i.e.*, a loop is an edge (v_i, v_j) where $v_i = v_j$.

In some directed as well as undirected graphs, we may have contain pair of nodes joined by more than one edges, such edges are called **multiple** or **parallel** edges. Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_f$ and $v_j = v_r$. Note that in case of directed edges, the two possible edges between a pair of nodes which are opposite in direction are considered distinct. So more than one directed edge in a particular direction in the case of a directed graph is considered parallel.

A graph which has neither loops nor multiple edges *i.e.*, where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph**. Fig. 13.2 (*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

Any graph which contains some multiple edges is called a **multigraph.** In a multigraph, no loops are allowed.

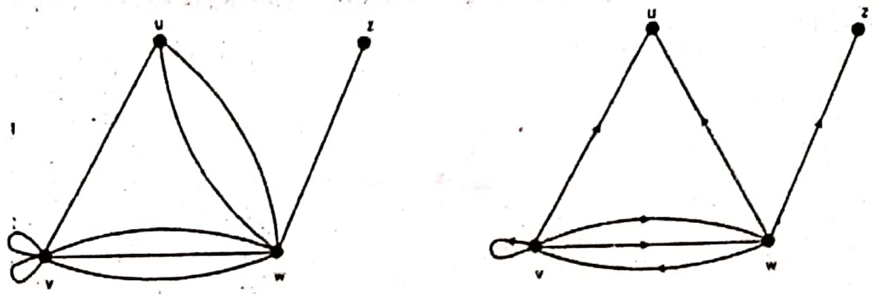


(a) Undirected multigraph (b) Directed multigraph

Fig. 13.7

If Fig. 13.7 (*a*) there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3. In Fig. 13.7. (*b*), there are two parallel edges associated with v_2 and v_3.

A graph in which loops and multiple edges are allowed, is called a **pseudograph.**



(a) Undirected pseudograph (b) Directed pseudograph

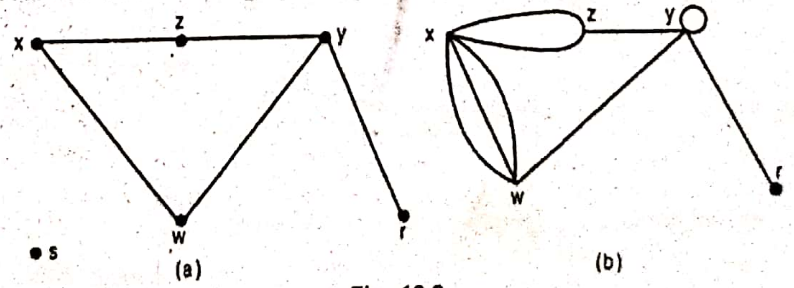
Fig. 13.7

It may be noted that there is some lack of standardisation of terminology in graph theory. Many words have almost obvious meaning, which are the same from book to book, but other terms are used differently by different authors.

13.4. Degree of a Vertex

The degree of a vertex of an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v in a graph G may be denoted by $\deg_G(v)$.

The degrees of vertices in the graph G and H in Fig. 13.9 are given below.



(a) (b)

Fig. 13.9

In G as shown in Fig. 13.9 (*a*) $deg_G(x) = 2 = deg_G(z) = deg_G(w), deg_G(y) = 3$ and $deg_G(r) = 1$ and in H as shown in Fig. 13.9 (*b*), $deg_H(x) = 5$, $deg_H(z) = 3$, $deg_H(y) = 5$, $deg_H(w) = 4$ and $deg_H(r) = 1$.

A vertex of degree 0 is called **isolated.** A vertex is **pendant** if and only if it has a degree 1. Vertex s in the graph G is isolated and vertex r is pendant. A vertex of a graph is called odd vertex or even vertex depending on whether its degree is odd or even.

In any graph G, we define

$$\delta(G) = \min\{\deg v : \in V(G)\} \text{ and}$$

$$\Delta(G) = \max\{\deg v : \in V(G)\}$$

If $v_1, v_2, \ldots, v_n$ are the n vertices of G, then the sequence $(d_1, d_2, \ldots, d_n)$ where $d_i = \deg(v_i)$ is the degree sequence of G. In general, we order the vertices so that the degree sequence is monotonically increasing *i.e.*

$$\delta(G) = d_1 \le d_2 \le \ldots \le d_n = \Delta(G).$$

For example, the degree sequence of the graph shown in Fig. 13.10.

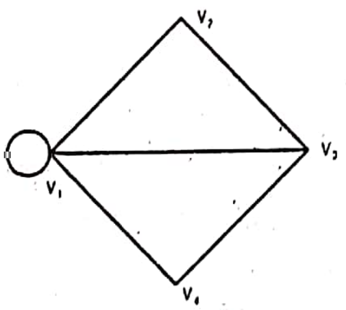


Fig. 13.10

is (2,2,3,5,) as $\deg(v_2) = \deg(v_4) = 2$, $\deg(v_3) = 3$ and $\deg(v_1) = 5$.

Theorem 13.1 A simple graph with at least two vertices has at least two vertices of same degree.

Proof. Let G be a simple graph with $n \ge 2$ vertices. The graph G has no loop and parallel edges. Hence the degree of each vertices is $\le n-1$. Suppose all the vertices of G are of different degrees. Hence the following degrees

$$0, 1, 2, 3, \ldots, n-1$$

are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree $n-1$ then v has $n-1$ adjacent vertices. Since v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree.

The converse of the above theorem is not true.

Theorem (the Handshaking theorem) 13.2. If G = (V, E) be an undirected graph with e edges. Then

$$\sum_{v \in V} \deg_G(v) = 2e$$

i.e., the sum of degrees of the vertices in an undirected graph is even.

Proof: Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees is equal twice the number of edges.

Note: This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary: In a non directed graph, the total number of odd degree vertices is even.

Proof: Let G = (V, E) a non directed graph. Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} \deg_G(v_i) = \sum_{v_i \in U} \deg_G(v_i) + \sum_{v_i \in W} \deg_G(v_i)$

$\Rightarrow \quad 2e - \sum_{v_i \in U} \deg_G(v_i) = \sum_{v_i \in W} \deg_G(v_i)$ ...(1)

Now $\sum_{v_i \in U} \deg_G(v_i)$ is also even

Therefore, from (1)

$\sum_{v_j \in W} \deg_G(v_i)$ is even

∴ Since for each $v_i \in W$, $\deg_G(v_i)$ is odd, the number of odd vertices in G must be even.

In degree and out degree

In a directed graph G, the out degree of a vertex v of G, denoted by $\text{outdeg}_G(v)$ or $\deg_G^+(v)$, is the number of edges beginning at v and the indegree of v, denoted by $\text{indeg}_G(v)$ or $\deg_G^-(v)$, is the number of edges ending at v. The sum of the in degree and out degree of a vertex is called the **total degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree is called a **sink**. Since each edge has an initial vertex and terminal vertex, the immediate theorem follows:

Theorem 13.3. If G = (V,E) be a directed graph with e edges, then

$$\sum_{v \in V} \deg_G^+(v) = \sum_{v \in V} \deg_G^-(v) = e$$

i.e., the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degrees of the vertices which equals the number of edges in G.

Proof : Any directed edge (u,v) contributes 1 to the in degree of u and 1 to the out degree of v. Further, aloop at v contributes 1 to the in degree and 1 to the out degree of v. Hence the proof.

In the directed graph G in Fig. 13.11.

$\text{Indeg}_G(a) = 2$, $\text{Indeg}_G(b) = 1$, $\text{Indeg}_G(c) = 2$, $\text{Indeg}_G(d) = 3$.

$\text{Outdeg}_G(a) = 1$, $\text{outdeg}_G(b) = 5$, $\text{outdeg}_G(c) = 1$, $\text{outdeg}_G(d) = 1$.

Note that, the sum of the in degrees and the sum of the out degrees each equal to 8, the number of edges.

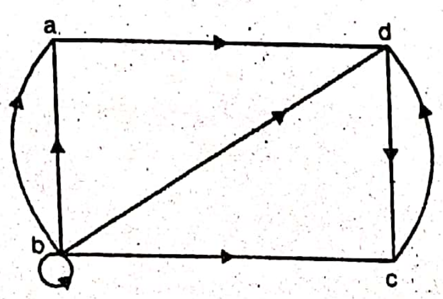
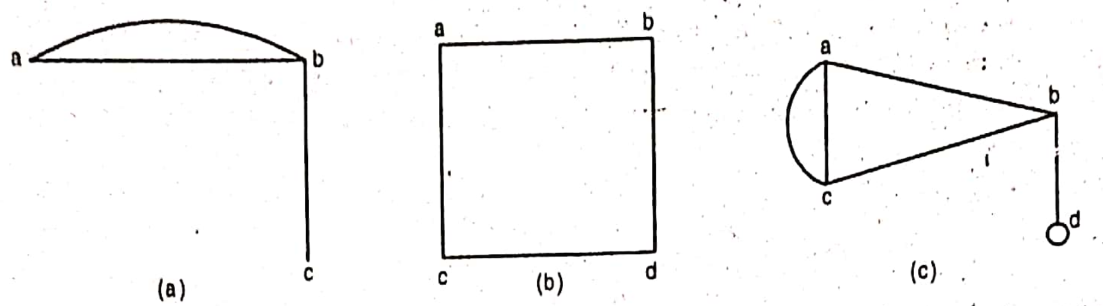


Fig. 13.11

Solved Examples

Example 1. State which of the following graphs are simple ?



(a) (b) (c)

Solution. (*a*) The graph is not a simple graph, since it contains parallel edge between two vertices a and b.

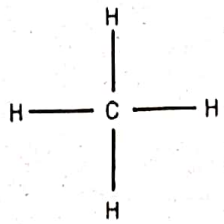
(*b*) The graph is a simple graph, it does not contain loop and parallel edge.

(*c*) The graph is not a simple graph since it contains parallel edge and a loop.

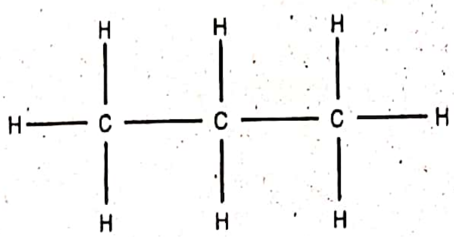
Example 2. Draw the graphs of the chemical molecules of

(*a*) methane (CH_4) (*b*) propane (C_3H_8)

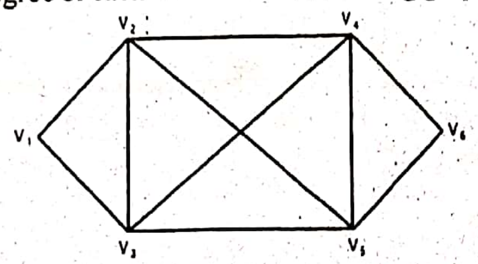
Solution. (*a*)



(*b*)



Example 3. Find the degree of each vertex of the following graph.

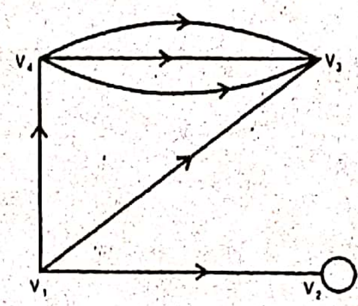


Solution. It is an undirected graph. Then

$\deg(v_1) = 2$, $\deg(v_2) = 4$, $\deg(v_3) = 4$

$\deg(v_4) = 4$ $\deg(v_5) = 4$, $\deg(v_6) = 2$

Example 4. Find the in degree out degree and of total degree of each vertex of the following graph.



Solution. It is a directed graph

in deg (v_1) = 0	out degree (v_1) = 3	total deg (v_1) = 3
in deg (v_2) = 2	out degree (v_2) = 1	total deg (v_2) = 3
in deg (v_3) = 4	out degree (v_3) = 0	total deg (v_3) = 4
in deg (v_4) = 1	out degree (v_4) = 3	total deg (v_4) = 4

Example 5. Show that the degree of a vertex of a simple graph G on n vertices can not exceed $n-1$.

Solution. Let v be a vertex of G, since G is simple, no multiple edges or loops are allowed in G. Thus v can be adjacent to at most all the remaining $n-1$ vertices of G. Hence v may have maximum degree $n-1$ in G. then $0 \leq \deg_G(v) \leq n-1$ for all $v \in V(G)$.

Example 6. Show that the maximum number of edges in a simple graph with n vertices is
$$\frac{n(n-1)}{2}$$

Solution. By the handshaking theorem
$$\sum_{i=1}^{n} d(v_i) = 2e,$$
where e is the number of edges with n vertices in the graph G.
$$\Rightarrow \quad d(v_1) + d(v_2) + \ldots\ldots d(v_n) = 2e \quad \ldots\ldots\ldots\ldots(1)$$
since we know that the maximum degree of each vertex in the graph G can be $(n-1)$. Therefore, equation (1) reduces
$$(n-1) + (n-1) + \ldots\ldots \text{ to } n \text{ terms} = 2e$$
$$\Rightarrow \quad n(n-1) = 2e \Rightarrow e = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Example 7. Is there a simple graph corresponding to the following degree sequences ?

(*i*) (1,1,2,3,)

(*ii*) (2,2,4,6,)

Solution. (*i*) Since the sum of degrees of vertices is odd, there exist no graph corresponding to this degree sequence.

(*ii*) Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exceed one less than the number of vertices.

Example 8. Does there exists a simple graph with seven vertices having degrees (1,3,3,4,5,6,6)?

Solution. Suppose there exists a graph with seven vertices satisfying the given properties. Since two vertices have degree 6, each of these two vertices is adjacent with every other vertex. Hence the degree of each vertex is at least 2, so that the graph has no vertex of degree 1 which is a contradiction. Hence there does not exist a simple graph with the given properties.

13.5. Types of Graphs

Some important types of graphs are introduced here. These graphs are often used as examples and arise in many applications.

Null Graph

A graph which contains only isolated node is called a **null graph** *i.e.*, the set of edges in a null graph is empty. Null graph is denoted on n vertices by N_n : N_4 is shown in Fig. 13.12. Note that each vertex of a null graph is isolated.

Fig. 13.12

Complete Graph

A simple graph G is said to be complete if every vertex in G is connected with every other vertex *i.e.*, if G contains exactly one edge between each pair of distinct vertices. A complete graph is usually denoted by K_n. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges. The graphs K_n for $n = 1, 2, 3, 4, 5, 6$ are shown in Fig. 13.13.

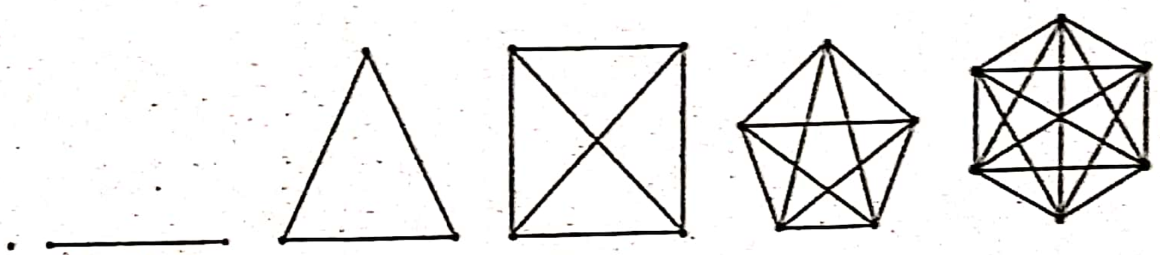


Fig. 13.13

Regular Graph

A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is r, then the graph is called a **regular graph of degree r**. Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n-1$. Also, note that if G has n vertices and is regular of degree r, then G has $(1/2)\, r\, n$ edges.

Example 9. What is the size of an r-regular (p, q)-graph ?

Solution: Since G is a r-regular graph, by the definition of regularity of G, we have, $\deg_G(v_i) = r$, for all $v_i \in V(G)$.

By the hand shaking theorem, $2q = \sum \deg_G(v_i)$

$$2q = \sum r = p \times r \Rightarrow \quad q = \frac{p \times r}{2}$$

Example 10. Does there exists a 4-regular graph on 6 vertices ? If so construct a graph.

Solution:

$$q = \frac{p \times r}{2} = \frac{6 \times 4}{2} = 12.$$

Four regular graph on 6 vertices is possible and it contains 12 edges. One of the graph is shown below.

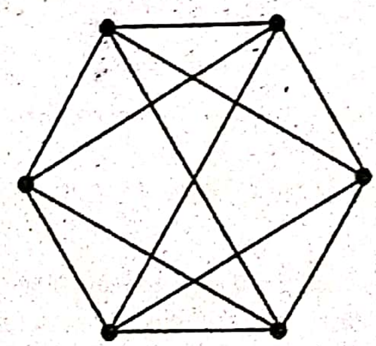


Fig. 13.17. The n-cube Q_n for n = 1,2 and 3.

Bipartite Graph

A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned ino two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G: Obviously, a bipartite graph can have no loop.

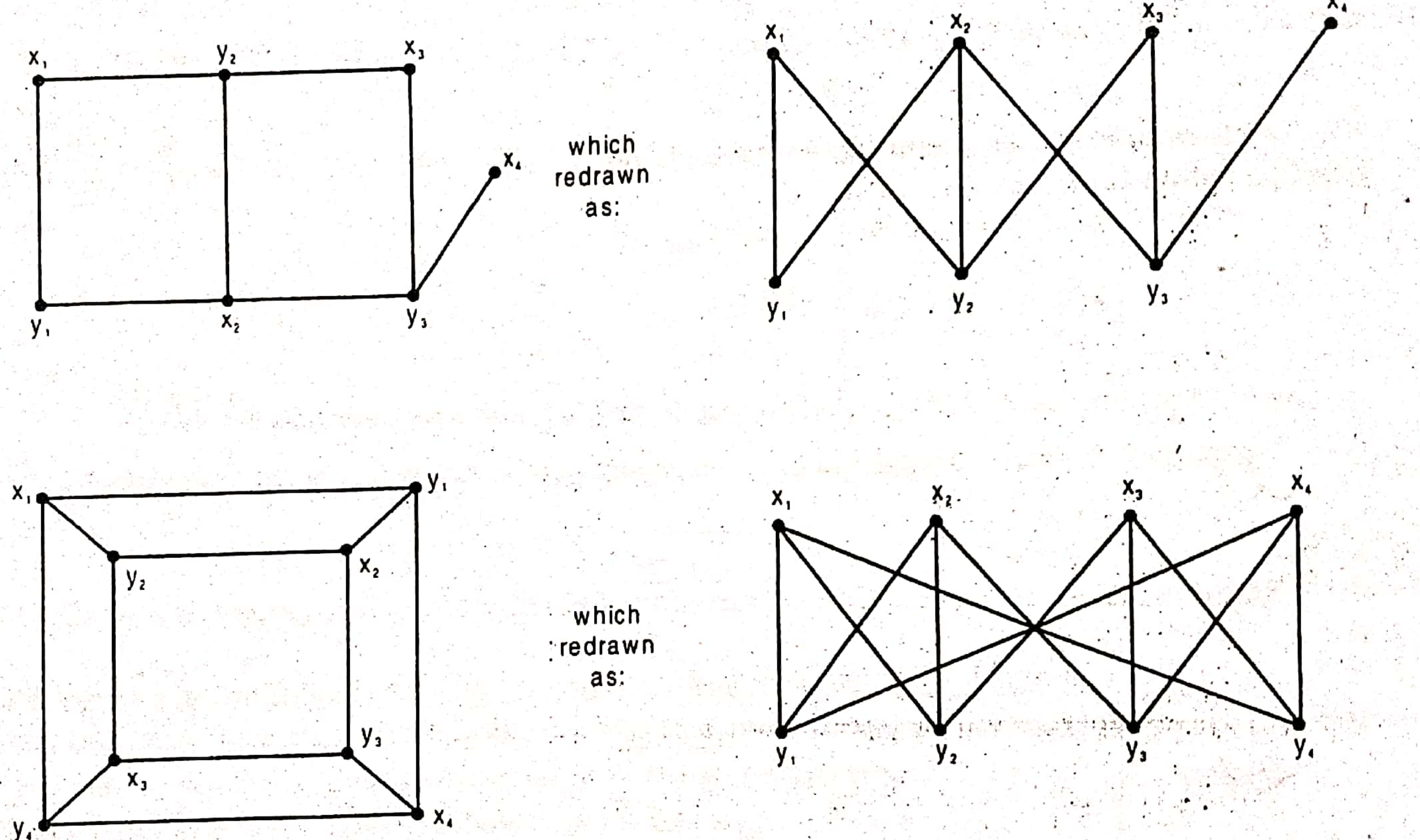


Fig. 13.18. Some bipartite graphs

Example 11. Show that C_6 is a bipartite graph.

Solution. C_6 is a bipartite graph as shown in Fig. 13.15 since its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

Complete bipartite graph

The complete bipartite graph on m and n vertices, denoted $K_{m,n}$ is the graph, whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices v_1 and v_6. Where v_1 is in V_1 and v_2 is in V_2. The complete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are shown in Fig. 13.19. Note that $K_{r,s}$ has $r + s$ vertices and rs edges.

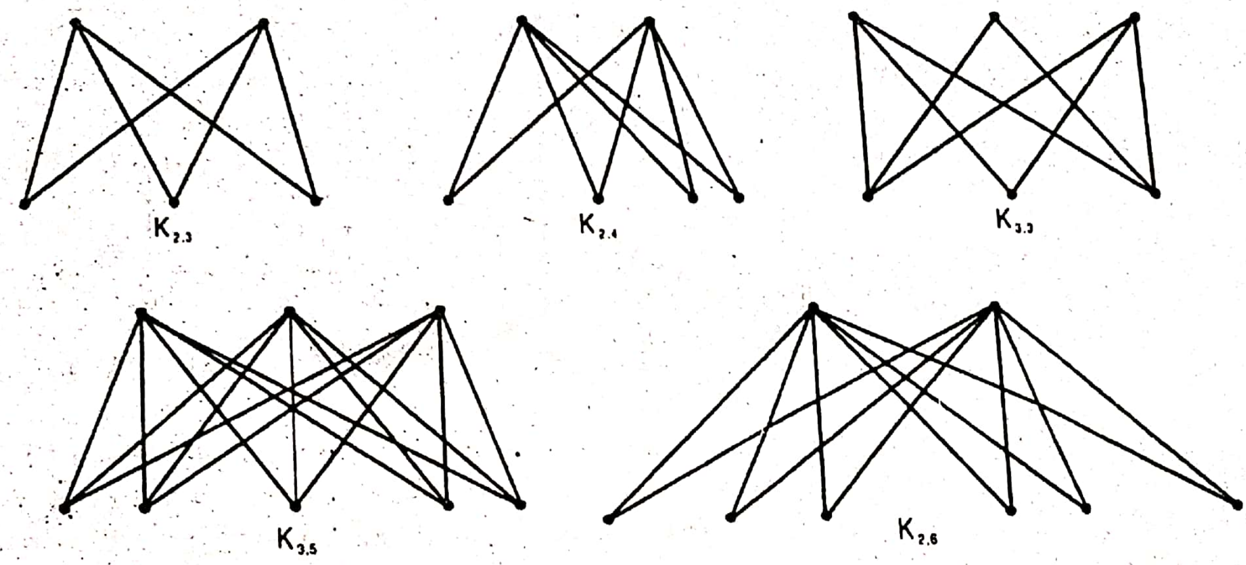


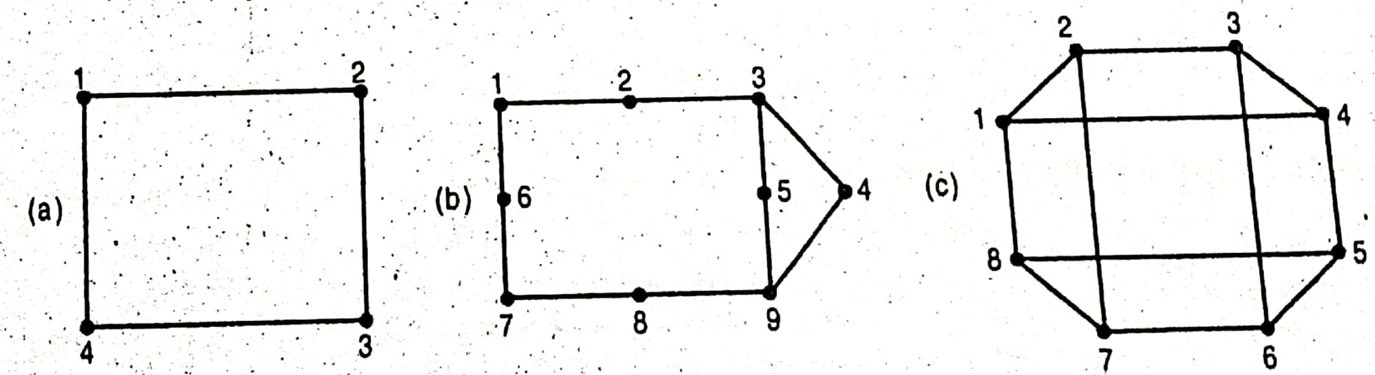
Fig. 13.19. Some complete bipartite graphs

A complete bipartite graph $K_{m,n}$ is not a regular if $m \neq n$.

Example 12. Prove that a graph which contains a triangle can not be bipartite.

Solution: At least two of the three vertices must lie in one of the bipartite sets since these two are joined by an edge, the graph can not be bipartite.

Example 13. Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.



Solution (*a*) The graph is not bipartite because it contains triangles (in fact two triangles)

(*b*) This is bipartite and the bipartite sets are {1,3,7,9,} and {2,4,5,6,8}

(*c*) This is bipartite and the bipartite sets are {1,3,5,7} and {2,4,6,8}

13.6 Subgraphs and Isomorphic Graphs

3. Draw a graph having the given properties or explain why no such graph exists.

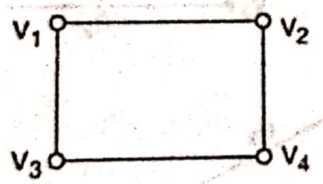
(a) Simple graph, four vertices each of degree two.

(b) Simple graph with five vertices having degrees 3, 3, 3, 3, 4.

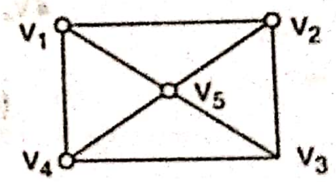
(c) Four edges; four vertices having degrees 1, 2, 3, 4.

Solution :

(a)



(b)



(c) Supppose there exists a graph G with the given properties. Then the sum of the degrees of the vertices of G is 10. In any graph we know that the sum of the degrees is twice the number of edges. Hence G must have five edges. Thus it follows that there does not exist any graph with the given properties.

4. Does there exist a simple graph with five vertices having degrees 2, 2, 4, 4, 4? Justify.

Solution : There does not exist any simple graph satisfying the given properties. Suppose there exists a simple graph with five vertices v_1, v_2, v_3, v_4, v_5 having degrees 2, 2, 4, 4, 4 respectively. Since the

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graph does not contain any loop and any parallel edges, it follows that each of the vertices v_3, v_4, v_5 must have four adjacent vertices. Hence v_1 is an adjacent vertex of v_3, v_4, v_5. This implies that the degree of v_1 is at least 3, which goes against the assumption. Hence there is no such graph.

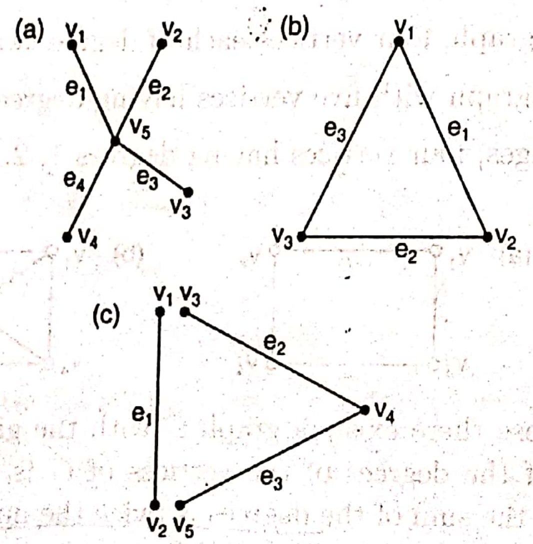
5. How many vertices are there in a graph with 15 edges if each vertex is of degree 3?

Solution : Let there be n vertices. Then the sum of the degrees of the vertices is $3n$. This sum is twice the number of edges. Hence $3n = 30$. So we find that the number of vertices is 10.

6. Show that a complete graph with n vertices consists of $n(n-1)/2$ edges.

Solution : Let G be a complete graph with n vertices. Let v be a vertex of G. Each of the remaining $n-1$ vertices is adjacent to v. Now G has no loop and parallel edges. Hence degree of each vertex is $n-1$. Thus the sum of the degrees of the vertices is $n(n-1)$ which is twice the number of total number of edges. Hence in a complete graph the number of edges is $n(n-1)/2$.

7. State which of the following graphs are bipartite graphs:



Solution : (a) The given graph is bipartite, since its vertex set is the union of two disjoint sets $\{v_1, v_2, v_3, v_4\}$ and $\{v_5\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset.

(b) This graph is not a bipartite graph. Suppose this graph is a bipartite graph. Then the vertex set is the union of two disjoint sets V_1 and V_2. Since the edge e_1 connects v_1 and v_2, we find that v_1 and v_2 cannot both belong to same set. Suppose $v_1 \in V_1$ and $v_2 \in V_2$.

Now e_2 joins v_2 and v_3. Hence v_3 must belong to V_1. Again e_3 joins v_3 and v_1. Since $v_3 \in V_1$, it follows that $v_1 \in V_2$. Thus we arrive at a contradiction. Hence the given graph is not a bipartite graph.

(c) The given graph is a bipartite graph, since its vertex set is the union of two disjoint sets $\{v_1, v_3, v_5\}$ and $\{v_2, v_4\}$ and each edge e joins a vertex in one of these sets to a vertex in the other subset.