Lecture 5: Test for Non-absolute convergence

Alternating series: A series with alternate positive and negative terms.

Example: $\sum (-1)^n$, $\sum \frac{(-1)^n}{n}$

Thm 5.1 (Alternating series test) Let $\sum (-1)^{n+1} z_n$ be an alternating series such that $\lim_{n\to\infty} z_n = 0$. Then the given series is convergent.

Proof Since

$$s_{2n} = z_1 - z_2 + z_3 - z_4 + \cdots + z_{2n-1} - z_{2n}$$

$$= (z_1 - z_2) + (z_3 - z_4) + \cdots + (z_{2n-1} - z_{2n})$$

and $z_k - z_{k+1} \geq 0$

therefore the subsequence $\langle s_{2n} \rangle$ of partial sums is increasing.

Now we have

$$s_{2n} = z_1 - (z_2 - z_3) - \cdots - (z_{2n-2} - z_{2n-1}) - z_{2n} \leq z_1 \quad \forall\, n \in \mathbb{N}$$

Hence by Monotone convergence theorem the subsequence $\langle s_{2n} \rangle$ converges to say $s \in \mathbb{R}$.

Now we show that the entire sequence $\langle s_n \rangle$ converges to s.

Let $\varepsilon > 0$. We choose $K \in \mathbb{N}$ such that $\forall\, n \geq K$

$$|s_{2n} - s| < \varepsilon/2 \quad \text{and} \quad |z_{2n+1}| < \varepsilon/2$$

Hence $\forall\ n \geq K$ we have

$$|s_{2n+1} - s| = |s_{2n} + z_{2n+1} - s|$$
$$= |s_{2n} - s + z_{2n+1}|$$
$$\leq |s_{2n} - s| + |z_{2n+1}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

~~Therefore we have~~

Hence we finally have $\forall\ \varepsilon > 0\ \ \exists\ K \in \mathbb{N}$ such that $n \geq K \Rightarrow$

$$|s_n - s| < \varepsilon$$

i.e. $\langle s_n \rangle$ converges to s

Hence the given series is convergent.

<u>Example</u> Consider the series $\sum \frac{(-1)^n}{n}$

Here $z_n = \frac{1}{n}$ and $\lim\limits_{n \to \infty} z_n = 0$

Hence this series is convergent.

The Dirichlet and Abel tests

~~Thm~~

Lemma 5.2 (Abel's lemma) (Partial summation formula) Let $\sum x_n$ and $\sum y_n$ be two infinite series. Let the partial sum of $\sum_{n=0}^{\infty} y_n$ be denoted by $\langle s_n \rangle$ with $s_0 = 0$. If $m > n$ then

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

Proof We have $y_k = s_k - s_{k-1}$ $\quad k = 1, 2, \dots$

Hence.
$$\sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$$

$$= \sum_{k=n+1}^{m} x_k s_k - \sum_{k=n+1}^{m} x_k s_{k-1}$$

$$= x_k s_m + \sum_{k=n+1}^{m-1} x_k s_k - x_{n+1} s_n - \sum_{k=n+2}^{m} x_k s_{k-1}$$

$$= (x_k s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} x_k s_k - \sum_{k=n+1}^{m-1} x_{k+1} s_k$$

$$= (x_k s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

Theorem 5.3 (Dirichlet test) Consider the two infinite series $\sum x_n$ and $\sum y_n$ such that $\lim\limits_{n\to\infty} x_n = 0$ and the sequence of partial sums of $\sum y_n$ is bounded. Then the series $\sum x_n y_n$ is convergent.

Proof Let $\langle s_n \rangle$ be the sequence of partial sums of $\sum y_n$.

~~By~~ We have let $|s_n| \leq B \quad \forall\, n \in \mathbb{N}$.

From Abel's lemma we have

$$\sum_{k=m+1}^{n} x_k y_k = (x_n s_n - x_{m+1} s_m) + \sum_{k=m+1}^{n-1} (x_k - x_{k+1}) s_k$$

$$\Rightarrow \left| \sum_{k=m+1}^{n} x_k y_k \right| \leq x_n |s_n| + x_{m+1} |s_m| + \sum_{k=m+1}^{n-1} |x_k - x_{k+1}| |s_k|$$

$$\leq x_n B + x_{m+1} B + \sum_{k=m+1}^{n-1} (x_k - x_{k+1}) B$$

$$= (x_n + x_{m+1}) B + \sum_{k=m+1}^{n-1} (x_k - x_{k+1}) B$$

$$= [x_n + x_{m+1} + x_{m+1} - x_n] B$$

$$= 2 x_{m+1} B$$

$$\Rightarrow \left| \sum_{k=m+1}^{n} x_k y_k \right| \leq 2 x_{m+1} B$$

Since $\lim\limits_{n\to\infty} x_k = 0$, Hence $\sum\limits_{k=m+1}^{n} x_k y_k$ is convergent by Cauchy criterion.

Theorem 5.4 (Abel's test) Suppose $\langle x_n \rangle$ is a convergent monotone sequence and the series $\sum y_n$ is convergent. Then the series $\sum x_n y_n$ is convergent.

Proof Suppose $\lim_{n\to\infty} x_n = x$ and $\langle x_n \rangle$ is a decreasing sequence

Let $u_n = x_n - x \quad \forall\, n \in \mathbb{N}$

Then $\lim_{n\to\infty} u_n = 0$ and u_n is also a decreasing sequence

Also we have

$$x_n = x + u_n$$

$$\Rightarrow x_n y_n = x y_n + u_n y_n$$

Since $u_n \to 0$ as $n \to \infty$ and $\sum y_n$ is convergent (hence its partial sums are bounded), therefore $\sum u_n y_n$ is convergent.

Also ~~since~~ $\sum x y_n$ is convergent.

Hence the given series $\sum x_n y_n$ is convergent

We can prove this fact similarly if $\langle x_n \rangle$ is increasing.