Tensor of Schaum's outline

③ Co-ordinate Transformation

Let for two reference frame ~~of~~ the co-ordinate of a be

$$\left.\begin{aligned} \bar{x}^1 &= \bar{x}^1(x^1, x^2 \ldots x^N) \\ \bar{x}^2 &= \bar{x}^2(x^1, x^2 \ldots x^N) \\ &\ldots\ldots \\ \bar{x}^N &= \bar{x}^N(x^1, x^2 \ldots x^N) \end{aligned}\right\} \text{—①}$$

by briefly.

$$\bar{x}^k = \bar{x}^k(x^1, x^2 \ldots x^N), \quad k = 1, 2, \ldots N \quad \text{—②}$$

For singled-valued, continuous derivation. The conversely be written as

$$x^k = x^k(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N), \quad k = 1, 2 \ldots N \quad \text{—③}$$

The relation ② or ③ define a transformation of coordinates from one frame to another.

(4) Contravariant and Co-variant vector

Let N quantity $A^1, A^2 \ldots A^N$ in $(x^1, x^2 \ldots x^N)$ system and $\bar{A}^1, \bar{A}^2 \ldots \bar{A}^N$ of $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N)$ system. Then by transformation eqns in summarised method

$$\bar{A}^P = \sum_{q=1}^{N} \frac{\partial \bar{x}^P}{\partial x^q} A^q \quad , \text{ where } P = 1, 2 \ldots N$$

Conventially writtem as

$$\bar{A}^P = \frac{\partial \bar{x}^P}{\partial x^q} A^q$$

This is the components of a contravariant vector or contravariant tensor of the 1st rank or first order. (see Problem 7.33 & 7.34).

On otherwise $A_1, A_2 \ldots A_N$ of N quantits of system $(x^1, x^2 \ldots x^N)$ and $\bar{A}_1, \bar{A}_2 \ldots \bar{A}_N$ of N quantits in $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N)$ system. Then transformation emertions

$$\bar{A}_P = \sum_{q=1}^{N} \frac{\partial x^q}{\partial \bar{x}^P} A_q \, , \quad P = 1, 2 \ldots N$$

simply written as

$$\bar{A}_P = \frac{\partial x^q}{\partial \bar{x}^P} A_q$$

known covariant vector or ~~con~~ covariant Tensor of the first rank or 1st order.

⑤ Contravariant, Covariant, and Mixed Tensor :-

Let N^2 quantities A^{qs} in $(x^1, x^2 \ldots x^N)$ system & $\bar{A}^{qs}$ in $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N)$ system

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}.$$

Simply. $\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$

Known as ~~Tens~~ Contravariant tensor of 2nd tor rank or rank two

My $\bar{A}_{pr} = \frac{\partial x^q \partial x^s}{\partial \bar{x}^p \partial \bar{x}^r} A_{qs}$

Covariant tensor of 2nd rank.

&

For Mixed Tensor

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

For 3rd $\bar{A}^{pr}_s = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^n} \frac{\partial x^s}{\partial \bar{x}^t} A^{qn}_t$, 3rd rank

) Kronecker Delta

$$\delta^j_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

(7) Tensor of Rank greater than two. ~~Tens~~

Any mixed tensor A^{qst}_{kl} $\bar{C}^{p}_{ij}$

$A^{qst}_{kl} \longrightarrow$ Rank — 5

Contravariant - order — 3

Covariant order — 2

Can be transform ate to the relation

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}$$

(8) Scalar or Invariant

$\phi = \bar{\phi}$ A . scalar or invarion known as

Tensor of rank zero

(9) Tensor field :- If any N-dimensional space If called define tensor any vector or scalar; It ~~should~~ be noted that a tensor or tensor field is not just the set of its components in one spacial coordinate system but all the possible sets under any ~~trans~~ transformation of co-ordinate.

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(10) Symmetric Tensor :-

No change due to two covariant indices or contravariant indices as →

$$A^{mpr}_{qs} = A^{pmr}_{qs} \quad \text{or} \quad A^{mpr}_{qs} = A^{mpr}_{sq}$$

All are symmetric Tensors.

(11) Anti symmetric Tensor

$$A^{mpr}_{qs} = -A^{pmr}_{qs} \quad \text{or.}$$

(12) Operation apply

(a) Addition

$$C^{mp}_{q} = A^{mp}_{q} + B^{mp}_{q}$$ (Commulative & Associative)

Same rank and type

(b) Substraction

$$D^{mp}_{q} = A^{mp}_{q} - B^{mp}_{q}$$

(c) Outer Multiplication

$$A^{pr}_{q} B^{m}_{s} = C^{prm}_{qs}$$

Summation of Contravariant & Summation covariant

(e) Inner Product :-

As outer product

$A^{mp}_{q} B^{r}_{st}$

putting $q = r$, $A^{mp}_{r} B^{r}_{st}$, putting $p = s$

another product

$A^{mp}_{r} B^{r}_{pt}$ another inner product is obtained.

(f) Quotient law :- Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor. This is known as quotient law.

(B) Matrix :
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Compact form $[a_{pq}]$, $p = 1 \cdots m$, $q = 1 \cdots n$

(14) Line Element and Matrix Tensor :

Line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, \partial u_p\, du_q$$

(Curvilinear Co-ord)

Known as Three 3D Euclidean space

generally for N - dimension

$$ds^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} g_{pq}\, du_p\, du_q$$

Conventionaly written as

$$ds = g_{pq}\, du_p\, du_q$$

Transformation of co-ordinate from x^j to $\bar{u}^k$

ch that ~~space~~ the metric form ~~into~~

formation into

$$(d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \dots + (d\bar{x}^N)^2 \text{ or}$$

$$d\bar{x}^k\, d\bar{u}^k$$

type coordinate Called

N- dimensional Euclidean space; in general known as "Riemanian"

(15) Conjugate or Reciprocal Tensor

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$, defined by g^{pq} as $g^{ij} = \frac{1}{g}$

$$g^{pq} = \frac{\text{Co-factor of } g_{pq}}{g}$$

This g^{pq} is a symmetric contravariant Tensor of rank Two ~~at~~ called the conjugate or reciprocal Tensor of g_{pq}. valid for

$$g^{pq} g_{rq} = \delta^{p}_{r} \quad //$$

(16) Associated Tensors

All tensors obtained from a given tensor by forming inner products with the metric tensor and its conjugate are called associate tensors of the given tensor.

eg - A^m and A_m are associated. tensor, the first are contravariant & 2nd covariant components. The relation between them is given by $A_p = g_{pq} A^q$ or $A^p = g^{pq} A_q$

For ~~the~~ Cartesian Cordinate system

$$g_{pq} = 1 \text{ if } p = q$$
$$= 0 \text{ if } p \neq q$$

So that $A_p = A^p$, no difference between contravariant & covariant components.

(17) Christoffel's Symbols :-

The following symbols

$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right)$$

&

$$\left\{ \begin{matrix} s \\ p\,q \end{matrix} \right\} = g^{sr}[pq, r]$$

Known as Christoffel's symbols of the first and 2nd kind. The 2nd kind $\left\{ \begin{matrix} s \\ p\,q \end{matrix} \right\}$ are $\{pq, s\}$ and Γ^{s}_{pq}

(18) Transformation law of Christoffel symbols

Suppose the co-ordinate $\bar{x}^k$ then

$$\overline{[jk, m]} = [pq, r] \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^m} + g_{pq} \frac{\partial x^p}{\partial \bar{x}^m} \cdot \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

$$\overline{\left\{ \begin{matrix} n \\ j\,k \end{matrix} \right\}} = \left\{ \begin{matrix} s \\ p\,q \end{matrix} \right\} \frac{\partial \bar{x}^n}{\partial x^s} \cdot \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial \bar{x}^n}{\partial x^q} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

are the laws of transformation of the Christoffel symbols showing that they are not tensors unless the terms on the right are zero