# Lec 2: Tests for convergence of series of non-negative terms 1

## Comparison tests

**Theorem 2.1** [Comparison test] Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be real sequences and suppose that for some $K \in \mathbb{N}$ we have

$$0 \leq x_n \leq y_n \qquad \forall\ n \geq K$$

(a) ~~Then~~ If $\sum y_n$ is convergent then $\sum x_n$ is convergent

(b) If $\sum x_n$ is divergent then $\sum y_n$ is divergent

Proof (a) Suppose $\sum y_n$ is convergent. Then for any $\varepsilon > 0$ $\exists$ $M(\varepsilon) \in \mathbb{N}$ such that

$$m > n \geq M(\varepsilon) \Rightarrow y_{n+1} + \cdots + y_m < \varepsilon \qquad [\text{Cauchy}]$$

If $n > \sup \{K, M(\varepsilon)\}$ then

$$0 \leq x_{n+1} + \cdots + x_m \leq y_{n+1} + \cdots + y_m < \varepsilon$$

This implies that $\sum x_n$ is convergent [Cauchy]

(b) Contrapositive of (a) i.e.

$$\sum x_n \text{ is not convergent} \Rightarrow \sum y_n \text{ is not convergent}$$

i.e. $\sum x_n$ is divergent $\Rightarrow$ $\sum y_n$ is divergent

<u>Thm 2.2 (Limit comparison test)</u> Suppose $\sum x_n$ and $\sum y_n$ are series with strictly positive terms and the following limit exists in $\mathbb{R}$:

$$r = \lim_{n\to\infty} \frac{x_n}{y_n}$$

(a) If $r \neq 0$ then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.

(b) If $r = 0$ ~~then~~ and if $\sum y_n$ is convergent then $\sum x_n$ is convergent.

<u>Proof</u> (a) Note that $r > 0$

Now we consider the sequence $\left\langle \frac{x_n}{y_n} \right\rangle$. Take $\varepsilon = \frac{r}{2} > 0$

Since $\frac{x_n}{y_n} \to r$ as $n \to \infty$, therefore $\exists\ K \in \mathbb{N}$ such that $\forall\ n \geq K$

$$\left| \frac{x_n}{y_n} - r \right| < \varepsilon = \frac{r}{2}$$

$$\Rightarrow -\frac{r}{2} < \frac{x_n}{y_n} - r < \frac{r}{2}$$

$$\Rightarrow \frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2} < 2r$$

Hence for $n \geq K$ we have

$$\left(\frac{1}{2} r\right) y_n \leq x_n < (2r)\, y_n$$

From first part of the inequality we have:

If $\sum x_n$ is convergent then $\sum y_n$ is convergent

From second part of the inequality we have:

If $\sum y_n$ is convergent then $\sum x_n$ is convergent

Hence $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent

(b) Since $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$

therefore $\forall\ \varepsilon > 0\ \exists\ K \in \mathbb{N}$ such that $n \geq K \Rightarrow$

$$\left|\frac{x_n}{y_n}\right| < \varepsilon \quad \text{i.e.} \quad \frac{x_n}{y_n} < \varepsilon$$

$$\Rightarrow \frac{x_n}{y_n} < 1 \qquad [\text{Taking } \varepsilon = 1]$$

$$\Rightarrow \cancel{\frac{x_n}{y_n}}\ x_n < y_n$$

Hence $\forall\ n \geq K$ we have

$$0 < x_n < y_n$$

Therefore if $\sum y_n$ is convergent then $\sum x_n$ is convergent.

Examples

1) The series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

Proof $\because$ $n^2+n > n^2 \quad \forall\ n \in \mathbb{N}$

Hence $0 < \frac{1}{n^2+n} < \frac{1}{n^2}$

$\because$ $\sum \frac{1}{n^2}$ converges, therefore $\sum \frac{1}{n^2+n}$ converges

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ is convergent

Proof Let $x_n = \frac{1}{n^2-n+1}$ $y_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} =$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2-n+1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}+\frac{1}{n^2}} = 1 \neq 0$$

$\because \sum y_n$ convergen, therefore $\sum x_n$ convergen

(c) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

Proof Let $x_n = \frac{1}{\sqrt{n+1}}$ $y_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = 1 \neq 0$$

$\because \sum y_n$ divergen, therefore $\sum x_n$ divergen.